# The (Not So) Simple Pendulum
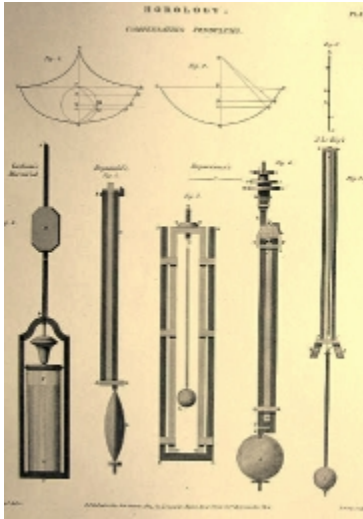
Pendulums are the defining feature of pendulum clocks, of course, but today they don't elicit much thought. Most modern "pendulum" clocks simply drive the pendulum to provide a historical look, but a great deal of ingenuity originally went into their design in order to produce highly accurate clocks. This essay explores horologic design efforts that were so important at one time—not gearwork, winding mechanisms, crutches or escapements (which may appear as later essays), but the surprising inventiveness found in the "simple" pendulum itself.

It is commonly known that Galileo (1564-1642) discovered that a swinging weight exhibits **isochronism**, purportedly by noticing that chandeliers in the Pisa cathedral had identical periods despite the amplitudes of their swings. The advantage here is that the driving force for the pendulum, which is difficult to regulate, could vary without affecting its period. Galileo was a medical student in Pisa at the time and began using it to check patients' pulse rates.

Galileo later established that the period of a pendulum varies as the square root of its length and is independent of the material of the pendulum bob (the mass at the end). One thing that surprised me when I encountered it is that the escapement preceded the pendulum—the verge escapement was used with hanging weights and possibly water clocks from at least the 14th century and probably much earlier. The pendulum provided a means of regulating such an escapement, and in fact Galileo invented the pin-wheel escapement to use in a pendulum clock he designed but never built. But it took the work of others to design pendulums for truly accurate clocks, and here we consider the contributions of three of these: Christiaan Huygens, George Graham and John Harrison.

It was Christiaan Huygens (1629-1695) who built the first pendulum clock as we know it on Christmas, 1656. His pendulum swung in a wide arc of about 30° and consisted of a metal ball suspended by silk threads. There are a few design aspects of pendulums that may appear obvious in retrospect but which were novel enough at the time. First, there is the matter of air and gear friction. To minimize these effects there must be sufficient mass to make frictional forces irrelevant, the rod of the pendulum should be thin, and the pendulum must be enclosed to avoid drafts. It is also true, unlike in Huygen's pendulum, that the bob itself should be thin—later bobs were made to slice through the air, and this feature along with the requirement for significant mass results in the tapered lens-shaped disk that we see today on pendulums.
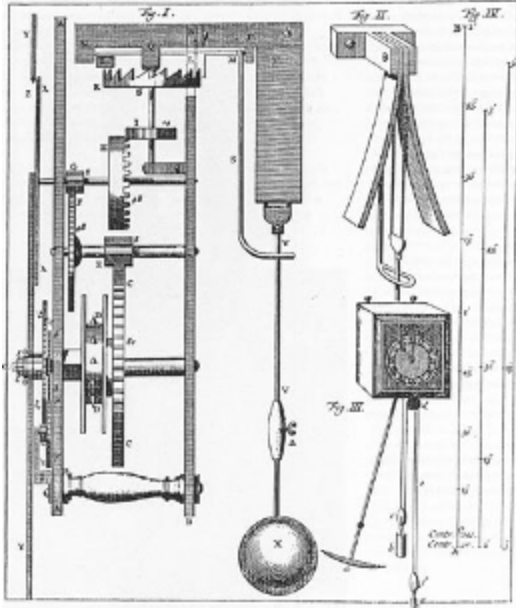
The second design feature we see in this clock design of Huygens is the concentration of mass at the bottom of the pendulum. While the mass itself does not contribute to the period of the pendulum (after frictional forces are overcome), the length of the pendulum certainly does, and this length is measured from the pivot point to the center of mass of the pendulum. The common "seconds" pendulum (which actually has a two-second period) is convenient for gearing ratios since most clock escapements move the seconds hands on each swing, and this pendulum has a length of a little over 39 inches. Huygens and Christopher Wren proposed this distance as a standard unit of length, but were unaware at that time of the geographic variation of this length (in 1793 the meter was standardized as one ten-millionth of the distance between the North Pole and equator based on curvature estimates from triangulations by Delambra and Mechain). Later measurements by Jean Richter of this geographic variation led Huygens to assign this variation to centripetal force from the Earth's rotation, which is indeed latitude dependent. The "seconds" length requirement makes for non-portable pendulum clocks, particularly if the center of mass of the pendulum is not as low as possible (and also exacerbated if the pendulum has a wide swing to accommodate as in Huygens' first clock).

Mahoney points out that Huygens' invention of the pendulum clock contained an important feature: the independent suspension of the pendulum and the crutch linked the clock mechanism and the pendulum, but in a way that allowed separate, one-way adjustments. The driving force of the escapement could be adjusted without affecting the operation of the pendulum, and varying the characteristics of the bob (such as increasing the bob mass to overcome variations in crutch coupling or streamlining it to decrease air resistance) did not affect the operation of the escapement. This allowed a practical means of calibrating the individual components. Also, the silk threads of the pendulum rod in this design were extremely light and strong with little stretch and high resistance to rot, and they also minimized friction at the pivot point. They were ideal for Huygens.



But the silk threads offered a huge advantage in one other way—they allowed the effective length of the pendulum to vary so the pendulum is truly isochronic. It had been discovered by Marin Mersenne (1588-1648) that the period of the pendulum does indeed depend on the amplitude of its swing, with isochronism true enough only for small angles. Wider swings produce slightly longer periods in the same proportion as $sin\ x$ varies from $x$ as the angle $x$ increases. This error is called the circular error or circular deviation. The wrapping of the silk threads through the "cheeks" seen in the clock diagram effectively shorten the pendulum length with its distance along its arc, constraining the pendulum to a **cycloidal** path and providing true

isochronism. Huygens revolutionized the design of pendulums through such mathematical analysis of this and other characteristics of pendulums.
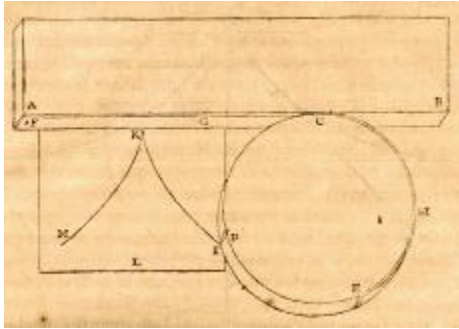


A cycloid is the curve defined by the path of a point on the edge of a circle as it rolls along a straight line, as shown in Huygens' figure on the left. It is a **tautochrone**, a curve for which a frictionless particle sliding on it under gravity to its lowest point will take the same amount of time regardless of its starting position on the curve. By definition an isochronic pendulum needs to follow a tautochronic path.

But let's back up a bit to see how Huygens found this curve. In deriving the relation for the period of a pendulum in 1659 equivalent to $T = 2\pi(L/g)^{1/2}$, Huygens found he had to make an approximation that was only negligible for small amplitudes of oscillation, one that defined a curved path of a cycloid with a vertical axis of half the pendulum length L. In one of those fortuitous circumstances that occur so frequently in history, he had studied precisely that curve for a mathematical challenge issued by Blaise Pascal in 1654. Following this lead Huygens found that a body falling from any point along the cycloid will reach the bottom in the same amount of time, and the ratio of this time to the time for free fall from rest along the axis of the cycloid is $\pi$ : 2. In 1673 he published his masterpiece, the *Horologium Oscillatorium*, in which he directly proved that the cycloid was the needed curve for an isochronous pendulum.

To derive this, Huygens begins by presenting the Galilean properties of free-fall, i.e., that the distance fallen is proportional to both the time squared and the velocity squared. In addition, the distance fallen in a given time is equal to the distance traversed in the same time with a constant velocity half that of the velocity at the end of the fall. After proving 20 more propositions too detailed to present here, he arrives at the figure to the right. Here the arc ABC is a cycloid created by point A as the circle AVD rolls along the top line DC. As translated from the Latin by Blackwell, Huygens states and then proves that:
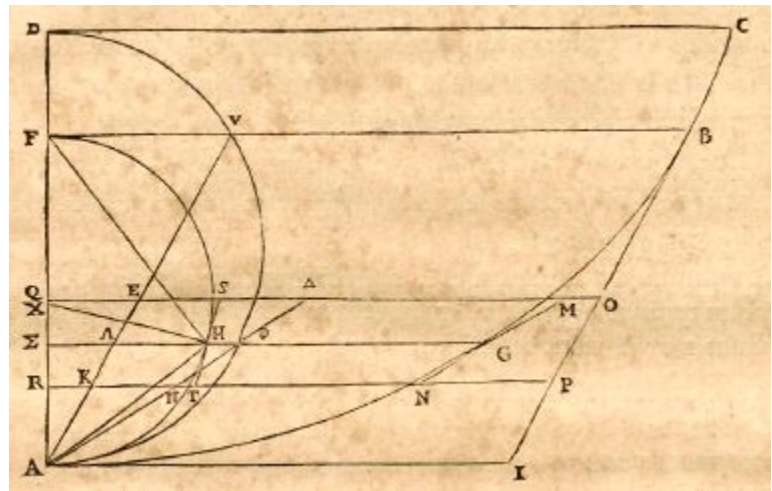


> The time in which a body crosses [spans] the line MN, with the uniform velocity acquired after it has fallen through the arc BG of the cycloid, will be related [proportional] to the time in which it would cross the line OP, with half of the uniform velocity which it would acquire by falling through the whole tangent BI, as the tangent ST is related to the part QR of the axis.

After one more proposition in which he considers infinitesimal arcs of travel along the cycloid (we'll touch on that later), Huygens arrives at the culmination of Part II of his work:

On a cycloid whose axis is erected on the perpendicular and whose vertex is located at the bottom, the times of descent, in which a body arrives at the lowest point at the vertex after having departed from any point on the cycloid, are equal to each other; and these times are related to the time of a perpendicular fall through the whole axis of the cycloid with the same ratio by which the semicircumference of a circle is related to its diameter.

where the last clause provides the $\pi : 2$ ratio mentioned earlier.

So an isochronous pendulum must be constrained to move along a cycloidal path. Huygens needed to find the curve (the **evolute**) that would "unwind" to form this cycloid (the **evolution** or **evolvent**), and he created a new branch of mathematics, the theory of evolutes, to do it. The problem reduced to finding a curve such that



- Each leaf is tangent to the centerline.
- Each leaf is perpendicular to the cycloidal arc of the pendulum at its point of contact.
- The leaf length to the point of contact with the bob equals the pendulum length, so it must have an arc length of twice the diameter of the circle generating the cycloidal path of the pendulum, which is half the cycloid measured from its base to vertex.

This leads to the fact that the evolute must be a curve having the same base, height and length as the cycloidal path of the pendulum, and Huygens came to the startling realization that the evolute is a cycloid generated by the same circle as the cycloid derived for the pendulum path, or in other words, the cycloid is its own evolute! (In 1692 Jacob Bernoulli showed that a logarithmic spiral also is its own evolute.)

But for a practical pendulum Huygens further proposed that it is necessary to know its "center of oscillation," and using an axiom equivalent to the conservation of energy he defined the center of gravity of a pendulum in terms of the modern concept of its moment of inertia. Taking the limits of infinitesimal points of mass, he calculated the centers of oscillation of many types of pendulums; for example, his spherical bob of radius $r$ on a weightless string produced a center of oscillation $2r^2/5L$ below the center. This provides the analysis of the effect of sliding weights on pendulums to adjust for (or measure) geographical differences. He derives the practical technique of using the period of a known pendulum to find $g$, the acceleration due to gravity of free-falling bodies. Finally, Huygens describes the conical pendulum and produces theorems on centrifugal force equivalent to (but preceding) Newton's $F=mv^2/r$. This is the only place where force appears, as his work is based on the concept of conservation of energy, still a popular approach to physical problems involving complicated motions.



The *Horologium Oscillatorium* was written in the style of geometric physics in which quantities are related by proportions

demonstrated with geometric constructions, a method soon superseded by the analysis tools of mathematical physics. But Huygens used infinitesimal time intervals and distances (see figure) and extrapolated them to limiting cases, prescient in his anticipation of the development of the calculus. Blackwell points out that while later physicists relied on the extensive foundations of calculus and mechanics to build arguments, Huygen's work "may be enjoyed as a beautiful specimen of [his] explicit handling of physical concepts and argument." It is a self-contained jewel with a brilliant clarity seen in great works of all fields.
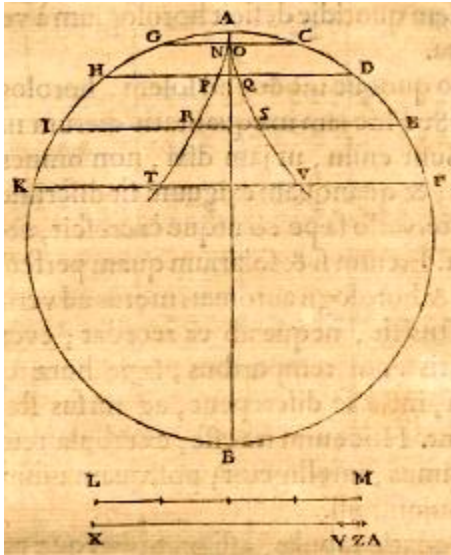


As an example of the practical nature of Huygens, he provides in this work an alternative method of drawing a cycloid that is shown in the figure on the left. Equal arcs AC, CD, and so forth are drawn on a circle whose diameter is half the length of the pendulum. Now join these points with horizontal lines and construct LM as the arc length AF, divided into as many parts as points marked off on AF. Then on the horizontal lines mark GO and CN with lengths equal to one part of LM, then HQ and DP with lengths equal to two parts of LM, etc. The curves connecting these points are the required cheek curves for the given length of the pendulum. To construct LM from the arc length AF, XZ is drawn equal to the sum of the two chords of half of AF. Then overlay XY as the length of the chord of AF. Finally add ZΔ as 1/3 of YZ. Then if AF is 1/6 or less of the circumference of the circle, XΔ equals AF to 1 part in 6000. What strikes me about this construction is that it creates an *approximate* solution, something I wouldn't expect from a geometric construction and surely an indication of how closely Huygens aligned his mathematics with the practical construction of mechanical clocks.

And maybe that's what makes the *Horologium Oscillatorium* such a fascinating piece of work. Mahoney points out something I hadn't noticed, that there are three layers of meaning in the diagrams and sketches of Huygens. In this work we see the overlay of physical shapes (the pendulum cord and cheeks) onto geometric constructions proving theorems about those shapes. In notes from 1659 in which Huygens first finds the cycloid as the isochronous curve, he also overlays an auxiliary curve, a parabola that describes the velocity of the bob as it moves along the cycloid. He created "a curve in physical space, the properties of whose normal and ordinate could be mapped by way of a mathematical curve so as to generate another mathematical curve congruent to a graph of velocity against distance" [Mahoney]. Later we will see that Huygens created a *mathematical relation* that defines isochronous systems, thereby lifting the mathematics out of the geometrical physics and anticipating analysis as the new physics.

Emmerson has provided analysis in his recent papers to show that a rigid pendulum suspended by a string between cheeks are not in fact isochronous, something derived as well by Benjamin Gompertz in 1818. And Huygens was aware of this. Emmerson provides a convincing argument that Huygens' statement in Part IV Proposition XXIV that it is not possible to determine the center of oscillation for pendulums (as opposed to bobs) suspended between cycloids was a realization that the center of mass approach was not a workable one for a rigid pendulum. "The matter is most difficult to explain…," Huygens says (from the Latin).

In this work Huygens also studied conical pendulums (in which the bob swings in a horizontal circle rather than in a vertical plane) as well as compound pendulums. He also designed pendulums for use aboard ships, a rocking platform for which pendulum clocks were never successfully produced. For this environment Huygens in 1672 created a clock that utilized a triangular pendulum, that is, one that is suspended from two separated cords and thereby constrained to move in one plane only, theoretically eliminating most effects of the rocking of the ship. The clock itself was suspended first in a ball-and-socket mount and later on gimbal mounts in an attempt to eliminate rocking in the plane of pendulum motion.
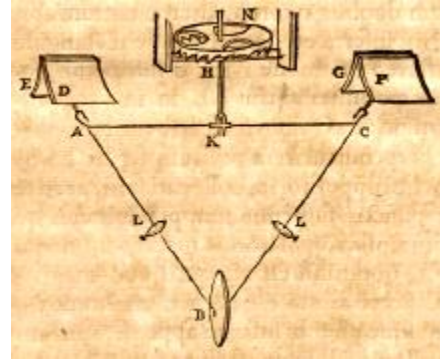
Huygens later invented the ingenious tri-cordal pendulum, a ring suspended at three points by threads and made to oscillate around its center as shown in his sketches below. Radial placement of weights could be used to calibrate the pendulum. From his analysis of conical pendulums he discovered that this mechanism would be isochronous if any point of the ring moves along a parabola curved around the cylinder defined by the ring. To fine-tune the tri-cordal pendulum to this constraint he considered adding cheeks but eventually just went with longer threads.
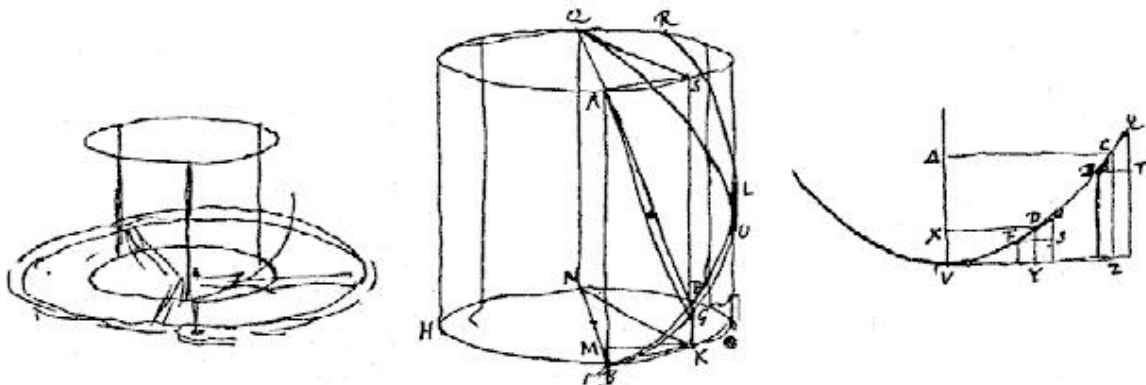


After the publication of his *Horologium Oscillatorium*, Huygens found that in a cycloidal pendulum the force on the bob is proportional to the distance or angle from the neutral position, and he deduced that any mechanical system that met this constraint would be isochronous. He came up with a number of mechanisms of this type. In 1675 this led him to invent the horizontal balance spring as a clock oscillator, in which the force varies directly with angle in the same way that force varies directly with distance in ordinary springs, although Hooke did not publish his law on this until 1678. (There is some debate today on whether Hooke actually invented the balance spring.)

But the elasticity of the balance spring suffered under temperature variations, and in 1693 Harrison revisited notes he made in 1684 to create the "Perfect Marine Balance." Here the spring balance was replaced with something like a physical pendulum, a balance bar on a knife edge swinging in a vertical
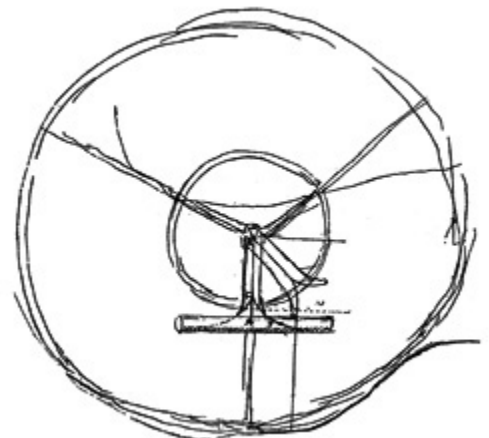
plane and controlled by another weight suspended by a thread from cycloidal cheeks that are mounted on the balance shaft and oscillate with the balance. To get the returning force to vary directly with the angle, he experimented with a weighted chain arrangement and with a float partly submerged in oil or mercury.

Meanwhile, in 1670 the anchor escapement was invented, possibly (and certainly claimed) by Robert Hooke. Some authors attribute its discovery to Thomas Tompion (1639-1713), but a more correct attribution may be to William Clement (1643-1710) [Heldman]. The workings of this escapement are outside the scope of this essay, but its effect on pendulum design was significant because it was used to reduce the pendulum swing to 4-5°. (It is worth noting here that the verge escapement can have as small an angle of escape as desired by designing very long pallet arms and a large distance between the horizontal escape wheel and the pivot arbor for the pallets and crutch—there are provincial French pendulum clocks, mostly of the 19th century, with this arrangment [Heldman]. But certainly the anchor escapement triggered clock designs at the time that had small-angle swings.)

A reduced pendulum swing makes possible much longer pendulums for a given horizontal space. Clocks with 14ft. pendulums were built, for example, and Tompion produced a clock with a 13ft. pendulum hung above the movement. Longer periods are more directly geared to clock time, but the small swing provided by the anchor escapement also significantly reduced friction at the pivot point. And now that the swing was small, the silk cords that rolled over the cycloidal cheeks were replaced with a short strip of flat metal (a brass suspension spring) that simply flexed around the shorter arc of the cheeks, decreasing the friction even more. When a metal rod and bob were connected to the strip, the entire pendulum manifested a permanent, all-metal construction. It might be noted here that Huygens and others looked to long, slow pendulums for stability, but in fact more success in pendulum clocks was ultimately had with short, fast-moving pendulums.

The other major problem with pendulums was the change in length, and therefore the center of gravity, with temperature. Huygens never fully realized the effect of temperature on his clocks. On hot days a pendulum lengthens slightly and the clock slows, and the opposite happens on cold days. George Graham (1673/4-1751) attempted to devise a pendulum using the varying expansion rate of metals to remain isochronous over temperature ranges. In these designs the expansion in temperature of one metal is offset by the expansion of the other, designed so that the net length of the pendulum remains constant. Failing to arrive at a suitable design, Graham settled on mercury-compensated pendulums as his solution. Here the pendulum is designed to hold mercury in a glass cylinder in much the same way as a mercury thermometer. When the pendulum length increased with temperature, the mercury expanded as well, and vice-versa. When properly designed, the net center of gravity of the pendulum remained unchanged regardless of temperature variations. Ingenious! Graham also invented the deadbeat escapement that made for quite small pendulum arcs.

(As an interesting notion Matthys mentions that if a pendulum is not temperature compensated, one might support the bob at the bottom edge. In this way the upward expansion of bob partially compensates for downward expansion of pendulum rod.)

John Harrison (1693-1776) approached Graham in 1730 with his invention of the "gridiron pendulum." Graham, a respected member of the Royal Society in London (and considered a humble, generous man according to all references I've seen), encouraged Harrison to pursue it, and it is the design seen in serious grandfather clocks (but faked in most commercial ones sold today). The design from a later clock is shown in the figure here. The expansion of brass rods in the pendulum compensates for the expansion of iron rods in order to keep the effective length the

same over temperature. The key is to have a high ratio between the thermal coefficients of the two metals used in the gridiron. If the ratio is 2:1, two rods can be used to expand downward and one rod upward, and so forth for different ratios. Anything less requires more rods to achieve the ratio. The ratio for iron and brass is 1.7:1, so Harrison used three rods expanding downward and two upward, and included two of these sets to equalize the weight. Since one rod can be common to both sets, nine rods were needed.

Well, just about. Harrison was also the first to realize the effect of atmospheric density on the period of a pendulum. Colder temperatures produce higher air densities, which alter the buoyancy of the bob and therefore the restoring torque. Another factor, absolute humidity, affects the density and viscosity and thence the rate of energy loss, equilibrioum amplitude and period of a pendulum [Emmerson]. From experiments performed with evacuated bell jars, Harrison adjusted his gridiron pendulum to account for the effect of temperature-induced density changes as well as for thermal expansion! Harrison also first confronted the effect of air resistance after he invented the grasshopper escapement—the frictional losses were now so low in his clock that the pendulum swung wildly until he attached small vanes to the pendulum. Air resistance is important, as over 90% of the drive energy imparted to a pendulum is lost through air drag. (Actually, some amount of air resistance can provide stability to the amplitude of the pendulum swing.) Through this and many other innovations Harrison claimed a pendulum clock accuracy of 1 second per month, an achievement still very much envied (and challenged).

There are other aspects of pendulums that are not considered here. For example, two or more pendulums that are lightly coupled (such as in clocks sitting on the same mantelpiece) will synchronize their swings, but in the opposite direction. Huygens made the first observation of a coupled oscillator in just this way in 1665 while recovering from an illness. For unconstrained simple pendulums with the same natural period, such a loose coupling results in a modal phenomenon in which the total swinging motion moves back and forth between them. Highly coupled oscillators, such as a compound pendulum where one pendulum is hung from the bob of another, exhibit chaotic motion. Coulomb used a torsion pendulum in 1784 to quantify the electrostatic force, and Cavendish determined the density of the Earth in 1798 using a pendulum. Foucault also famously used a pendulum in 1851 to directly demonstrate the rotation of the Earth. But in the end my fascination lies with the creative, technical pursuits seen in the early designs of pendulum clocks.

**References**

Andrewes, W. J. H. (Ed.). **The Quest for Longitude: The Proceedings of the Longitude Symposium Harvard University, Cambridge, Massachusetts, November 4-6, 1993.** Cambridge: Collection of Historical Scientific Instruments, Harvard University (1996). Wow, is this a neat book, a lavishly illustrated collection of fascinating essays by experts in horology on the pursuit to determine the longitude of a person at sea, a huge historical problem. Essays that provided information for the present essay include *The Longitude Timekeeper of Christiaan Huygens*, by J.H. Leopold; *'John Harrison, Clockmaker &amp; Barrow; Near Barton upon Humber; Lincolnshire': The Wooden Clocks, 1713-1730*, by Andrew L. King; and *The Scandalous Neglect of Harrison's Regulator Science*, by Martin Burgess.

Emmerson, Alan. The papers in Horological Science by Mr. Emmerson at http://users.qldnet.com.au/~ajay/From%20an%20Engineer's%20Notebook/ENIndex.html are a

pedagogical treat, presenting clear mathematical explanations of pendulum physics such as the non-isochronous behavior of a rigid pendulum suspended between cheeks as mentioned in this essay.

Heldman, Alan W. *Personal Communications*. Mr. Heldman's horological knowledge led to several corrections and improvements to this essay, which is much appreciated.

Huygens, Christiaan. **The Pendulum Clock or Geometrical Demonstration Concerning the Motion of Pendula as Applied to Clocks, Translated with Notes by Richard J. Blackwell.** Ames: Iowa State University Press (1986 translation of 1673 *Horologium Oscillatorium*). Surprisingly, this is the first English translation of Huygens' book, and it's a really interesting read. This is also the culminating scientific work presented as geometrical physics (i.e., using geometric constructions as derivations and proofs, with relations between quantities of different dimensions expressed as proportions rather than equations). Later works by others trended toward analytical approaches, particularly following the invention of calculus. Interestingly, Blackwell also notes that this book is based on an axiom equivalent to the conservation of energy rather than the concept of forces developed later by Newton. An actual scan of the 1673 book from which the manuscript figures of this essay were drawn is found at http://kinematic.library.cornell.edu:8190/kmoddl/toc_huygens1.html

King, Henry C. **Geared to the Stars: The Evolution of Planetariums, Orreries, and Astronomical Clocks.** Toronto: University of Toronto Press (1978). An encyclopedic, out-of-print work on a niche subject that is quite an expensive volume to buy on the used market. I was able to borrow it from a local library.

Mahoney, Michael S. Various fascinating papers, most of which involve Huygens, can be found at http://www.princeton.edu/~mike/17thcent.html. In particular, details of Huygens' original cycloidal derivations from 1659 can be found in *Christiaan Huygens: The Measurement of Time and Longitude at Sea*, and an interesting discussion of the physical and mathematical layers within Huygens' drawings is presented in *Drawing Mechanics*.

Matthys, Robert J. **Accurate Clock Pendulums.** Oxford University Press (2004). Lots of practical advice can be found in this book.