The Art of Nomography

Dead Reckonings: Lost Art in the Mathematical Sciences
http://www.myreckonings.com/wordpress

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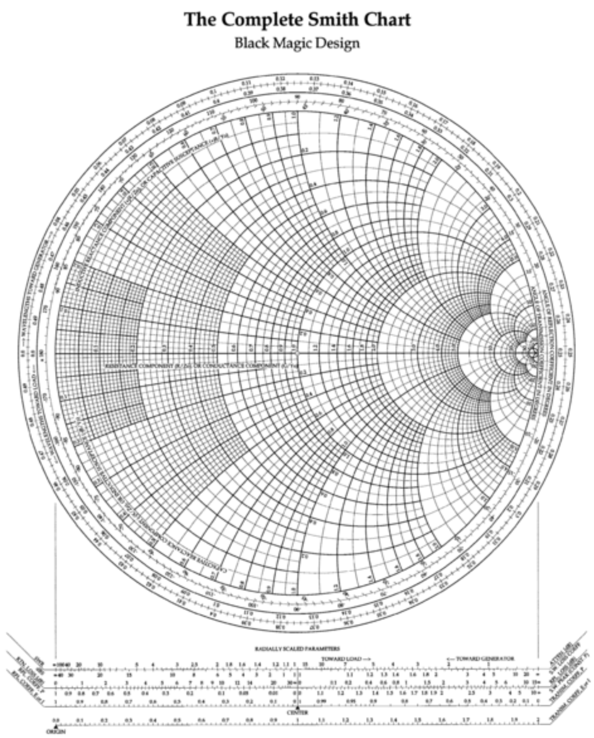
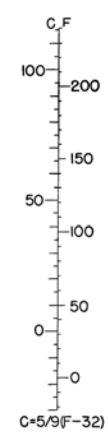


Nomography, truly a forgotten art, is the graphical representation of mathematical relationships or laws (the Greek word for law is *nomos*). These graphs are variously called **nomograms** (the term used here), **nomographs**, **alignment charts**, and **abacs**. This area of practical and theoretical mathematics was invented in 1880 by Philbert Maurice d'Ocagne (1862-1938) and used extensively for many years to provide engineers with fast graphical calculations of complicated formulas to a practical precision. Along with the mathematics involved, a great deal of ingenuity went into the design of these nomograms to increase their utility as well as their precision. Many books were written on nomography and then driven out of print with the spread of computers and calculators, and it can be difficult to find these books today even in libraries. Every once in a while a nomogram appears in a modern setting, and it seems odd and strangely old-fashioned—the multi-faceted Smith Chart for transmission line calculations is still sometimes observed in the wild. The theory of nomograms "draws on every aspect of analytic, descriptive, and projective geometries, the several fields of algebra, and other mathematical fields" [Douglass]. This essay is an overview of how nomograms work and how they are constructed from scratch. The first part of this essay is concerned with these straight-scale designs, the second part addresses nomograms having one or more curved scales, and the third part describes how nomograms can be transformed into different shapes.



The simplest form of nomogram is a scale such as a Fahrenheit vs. Celsius scale seen on an analog thermometer or a conversion chart. Linear spacing can be replaced with logarithmic spacing to handle conversions involving powers. Slide rules also technically qualify as nomograms but are not considered here. A slide rule is designed to provide basic arithmetic operations so it can solve a wide variety of equations with a sequence of steps, while the traditional nomogram is designed to solve a specific equation in one step. It's interesting to note that the nomogram has outlived the slide rule.

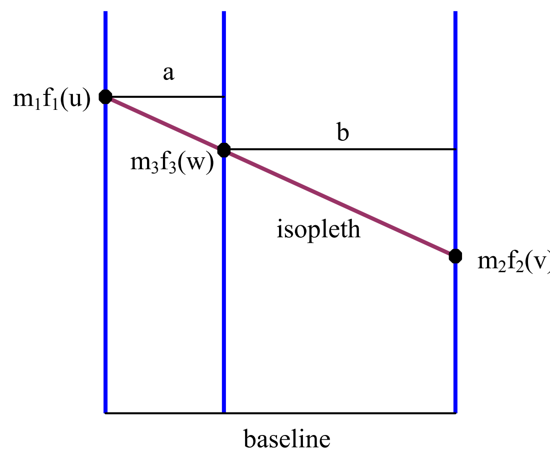
Most of the nomograms presented here are the classic forms consisting of three or more straight or curved scales, each representing a function of a single variable appearing in an equation. A straightedge, called an **index line** or **isopleth**, is placed across these scales at known values of these variables, and the value of an unknown variable is found at the point crossed on that scale. This provides an analog means of calculating the solution of an equation involving one unknown, and for finding one variable in terms of two others it is much easier than trying to read a 3-D surface plot. We will see later that it is sometimes possible to overlay scales so the number of scale lines can be reduced.

Geometric Design

We can design nomograms composed of straight scales by analyzing their geometric properties, and a variety of interesting nomograms can be constructed from these derivations. Certainly these seem to be the most prevalent types of nomograms.

Parallel Scale Nomograms

The figure on the right shows the basic parallel scale nomogram for calculating a value $f_3(w)$ as the sum of two functions $f_1(u)$ and $f_2(v)$:



$$f_1(u) + f_2(v) = f_3(w)$$

Each function plotted on a vertical scale using a corresponding scaling factor (sometimes called a scale modulus) m1, m2 or m3 that provides a conveniently sized nomogram. The spacing of the lines is shown here as a and b. Now by similar triangles, $[m_1f_1(u) – m_3f_3(w)] / a = [m_3f_3(w) – m_2f_2(v)] / b$. This can be rearranged as:

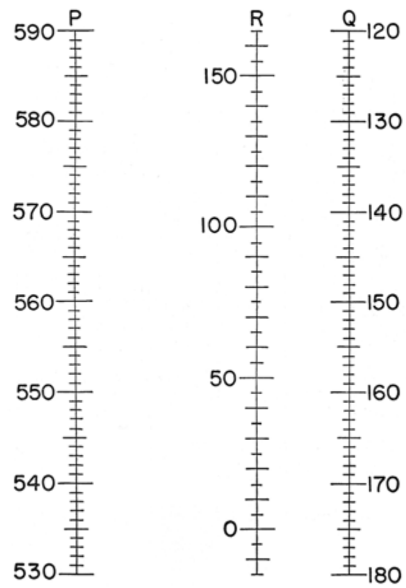
$$m_1f_1(u) + (a/b)\, m_2f_2(v) = (1 + a/b)\, m_3f_3(w)$$

So to arrive at the original equation $f_1(u) + f_2(v) = f_3(w)$, we have to cancel out all the terms involving m, a and b, which is accomplished by setting $m_1 = (a/b)\, m_2 = (1 + a/b)m_3$. The left half of this relationship provides the relative scaling of the two outer scales and the outer parts provide the scaling of the middle scale:

$$m_1 / m_2 = a / b \qquad m_3 = m_1m_2 / (m_1 + m_2)$$

Also note that the baseline does not have to be perpendicular to the scales for the similar triangle proportion to be valid.

Now a = b for the case where the middle scale is located halfway between the outer scales, and in this case $m_1 = m_2$ and $m_3 = ½\, m_1$. For a smaller range and greater accuracy of an outer scale, we can change its scale m and move the middle line away from it and toward the other outer scale. In fact, if the unknown scale w has a very small range it can be moved *outside* the two other scales to widen the scale. Additions to u, v or w simply shift the scale values up or down. Multipliers of u, v and w multiply the value when drawing the scales (they are not included in the values of m in the above calculations). Subtracting a value simply reverses the up/down direction of the scale, and if two values are negative their scales can simply be swapped. The example on the right shows a parallel-scale nomogram for the equation **(u–425) – 2(v–120) = w** designed for ranges $530<u<590$ and $120<v<180$.



(P−425)−2(Q−120)=R

So this looks like a lot of work to solve a simple linear equation. But in fact plotting logarithmic rather than linear scales expands the use of parallel scale nomograms to very complicated equations! The use of logarithms allows multiplications to be represented by additions and powers to be represented by multiplications according to the following rules:

$$\log (cd) = \log c + \log d \qquad \log c^d = d \log c$$

So if we have an equation such as $f_1(u) \times f_2(v) = f_3(w)$, we can replace it with

$$\log [f_1(u) \times f_2(v)] = \log f_3(w)$$

or,

$$\log f_1(u) + \log f_2(v) = \log f_3(w)$$

and we have converted the original equation into one without multiplication of variables. And note that there is actually no need to solve symbolically for the variable (we just plot these logs on the scales), a significant advantage when we come to more complicated equations.

Let's create a nomogram for the engineering equation $\mathbf{(u + 0.64)^{0.58}(0.74v) = w}$ as given in Douglass. We assume that the engineering ranges we are interested in are $1.0<u<3.5$ and $1.0<v<2.0$.

$$0.58 \log (u + 0.64) + \log (0.74v) = \log w$$

$$0.58 \log (u + 0.64) + \log (0.74) + \log v = \log w$$

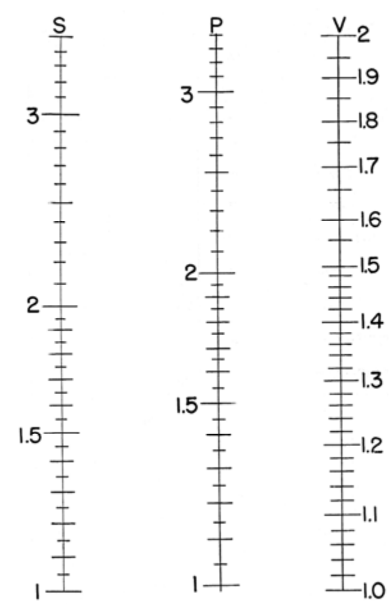
$$0.58 \log (u + 0.64) + \log v = \log w - \log (0.74)$$

We will directly plot the three components here as our u, v and w scales. To find the scaling factors we divide the final desired height of the u and v scales (say, 6 inches for both) by the ranges (maximum – minimum) of u and v:

$$m_1 = 6 / [0.58 \log (3.5 + 0.64) - 0.58 \log (1 + 0.64)] = 25.72$$

$$m_2 = 6 / [\log 2.0 - \log 1.0] = 19.93$$

$$m_3 = m_1 m_2 / (m_1 + m_2) = 11.23$$



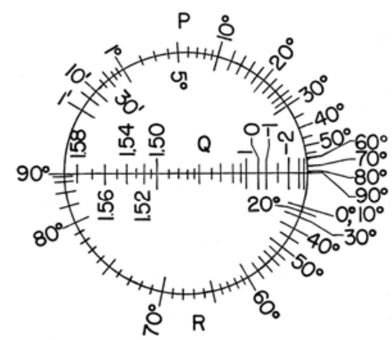
$(S+0.64)^{.58}(0.74V)=P$

Let’s set the width of the chart to 3 inches:

$a / b = m_1 / m_2 = 1.29$ so $a = 1.29b$

$a + b = 3$ so $1.29b + b = 3$ yielding $b = 1.31$ inches and $a = 1.69$ inches

We draw the u-scale on the left marked off from u = 1.0 to u = 3.5. To do this we mark a baseline value of 1.0 and place tick marks spaced out as 25.72 [0.58 log (u + 0.64) – 0.58 log (1.0 – 0.64)] which will result in a 6 inch high line. Then 3 inches to the right of it we draw the v-scale with a baseline value of 1.0 and tick marks spaced out as 19.93 (log v – log 1). Finally, 1.69 inches to the right of the u-scale we draw the w-scale with a baseline of $(1.0 + 0.64)^{0.58}(0.74)(1) = 0.98$ and tick marks spaced out as 11.23 (log w – log(0.74). And we arrive at the nomogram on the right, where a straightedge connecting values of u and v crosses the middle scale at the correct solution for w, and in fact any two of the variables will generate the third. Flexibility in arranging terms of the equation into different scales provides a means of optimizing the ranges and accuracies of the nomogram. A larger scale and finer tick marks can produce a quite accurate parallel scale nomogram that is deceptively simple in appearance, and one that can be manufactured and re-used indefinitely for this engineering equation.

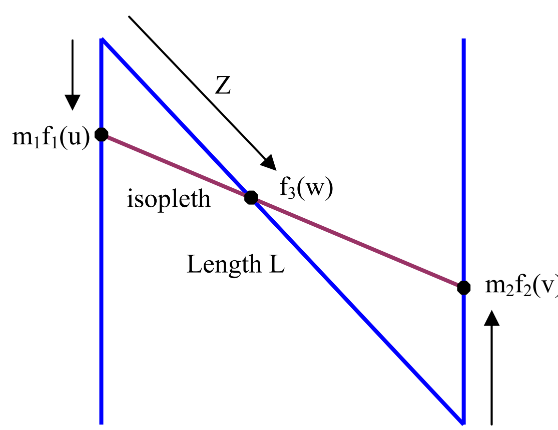


$\log\sin P(Q^3-4)=7\cos^2 R$

It is also possible to create a circular nomogram to solve a 3-variable equation. Details on doing this from geometrical derivations are given in Douglass.

N or Z Charts

A nomogram like that shown in the figure on the right is called an “N Chart” or more commonly a “Z Chart” because of its shape. The slanting middle scale joins the baseline values of the two outer scales (which are now plotted in opposition). The middle line can slant in either direction by flipping the diagram, and it can be just a partial section anchored at one end or floating in the middle if the entire scale isn’t needed in the problem, thus appearing, as Douglass puts it, “rather more spectacular” to the casual observer. A Z chart can be used to solve a 3-variable equation involving a division:

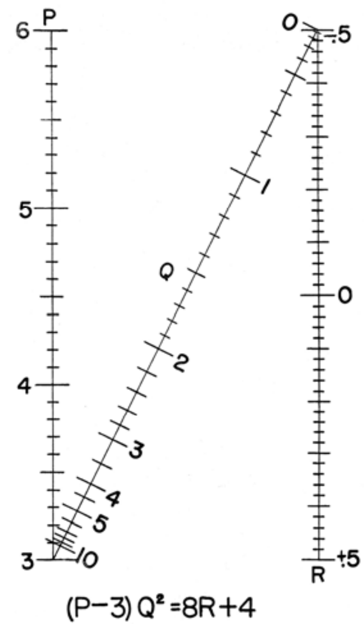


$f_3(w) = f_1(u) / f_2(v)$

By similar triangles, $m_1f_1(u) / m_2f_2(v) = Z / [L – Z]$. Substituting $f_3(w)$ for $f_1(u) / f_2(v)$ and rearranging terms yields the distance along Z for tick marks corresponding to $f_3(w)$:

$$Z = L\, f_3(w) / [(m_2/m_1) + f_3(w)]$$

The $f_3(w)$ scale does not have a uniform scaling factor m_3 as before. We could have used a parallel scale chart with logarithmic scales to plot this division, but the Z chart performs this with linear scales for u and v and it was once a real chore to calculate logarithms. But further, the linear scales of the Z Chart are much more suitable for combining a division with an addition or subtraction than compound parallel scales with their logarithmic scales. And of course if the scale for the unknown variable is an outside one, we have a Z chart for multiplication.

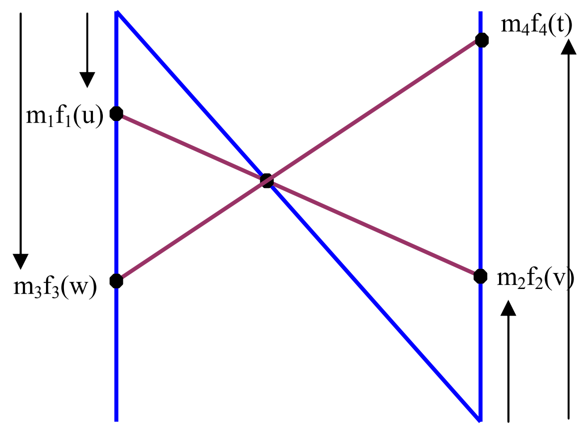


An example of a Z chart is shown here for the equation $\mathbf{Q^2 = (8R+4) / (P-3)}$. To create this, the desired height of the nomogram and the ranges of P and R provide their scaling factors m_1 and m_2 as done earlier. The desired width of the chart along with this height defines the length L needed for the Q-scale ($L^2 = W^2 + H^2$). The tick marks for Q are located a distance from the end calculated from the formula for Z above, where $f_3(w)$ is replaced with Q^2. It is also possible to slide the outer scales up or down without changing the tick mark spacing of the Z-scale as it also rotates due to its endpoints (because similar triangles still result), yielding in a nomogram with a perpendicular Z-scale as shown in an example in the second part of this essay.

Proportional Charts

The proportional chart solves an equation in four unknowns of the type

$$f_1(u) / f_2(v) = f_3(w) / f_4(t)$$

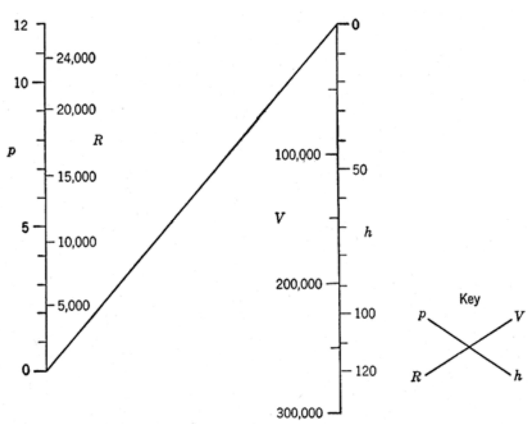


If we take our Z chart diagram and a second isopleth that intersects the Z line at the same point as the first, we have by similar triangles:

$$m_1 f_1(u) / m_2 f_2(v) = m_3 f_3(w) / m_4 f_4(t)$$

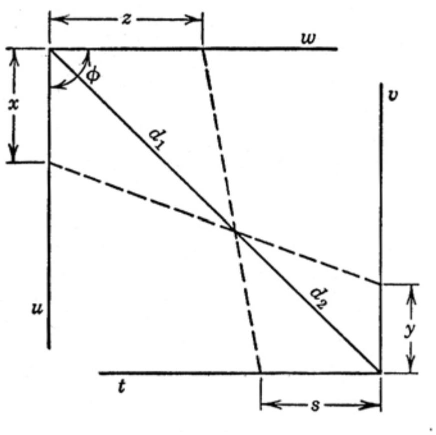
which matches our equation above if we choose the scaling of the outer scales such that

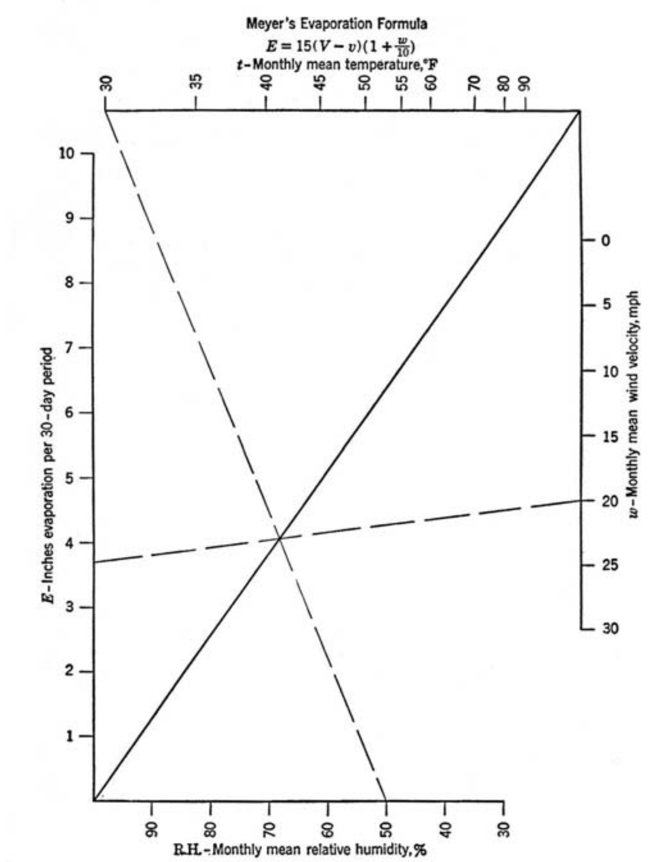
$$m_1 / m_2 = m_3 / m_4$$



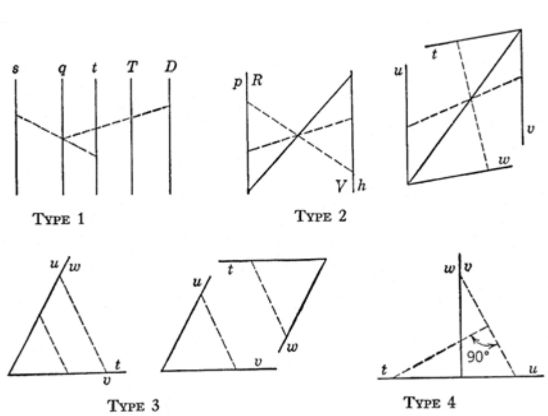
We then overlay two variables on each outer scale with this ratio of scaling factors, as shown in the nomogram to the right from Josephs for the approximate pitch of flange rivets in a plate girder, where p is the rivet pitch in inches, R is the rivet value in lbs, h is the effective depth of the girder in inches, and V is the total vertical shear in lbs: $p = Rh/V$.

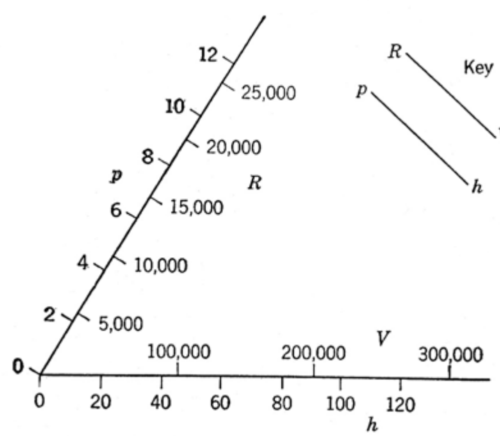
Another type of proportional chart uses crossed lines within a boxed area, as shown below. Again, the scaling factors for the four variables are given by $m_1 / m_2 = m_3 / m_4$ where these are related as before to the u, v, w and t scales, respectively. (Actually, similar triangles still exist and the ratios still hold for any parallelogram, not just a rectangle.)





But there are other types of proportional charts as shown below. In the ones labeled Type 3 an isopleth is drawn between two scale variables, then moved parallel until it spans the third variable value and the fourth unknown variable. The flange rivet example done in this manner is shown here. In the Type 4 nomogram the second isopleth is drawn perpendicular rather than parallel to the first one.



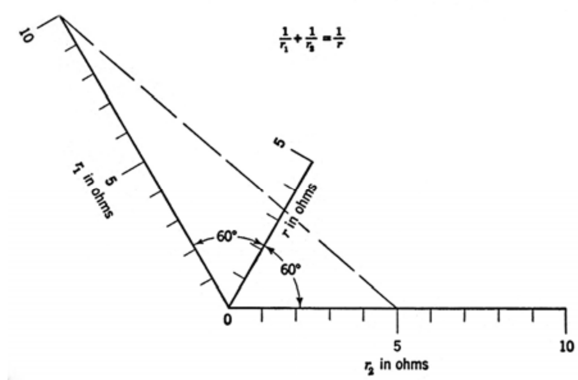


Concurrent Scale Charts

The concurrent chart solves an equation of the type

$$1/f_1(u) + 1/f_2(v) = 1/f_3(w)$$

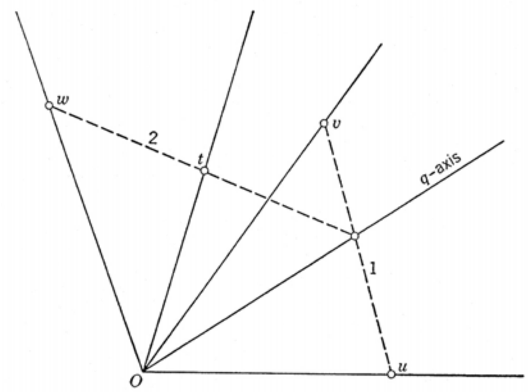
The effective resistance of two parallel resistors is given by this equation, and a concurrent scale nomogram for this is shown on the right.



The derivation is somewhat involved, but in the end the scaling factors m must meet the following conditions:

$$m_1 = m_2 = m_3 / (2 \cos A)$$

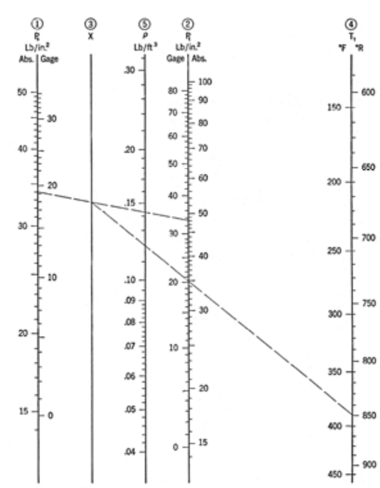
where A is the angle between the u-scale and the v-scale, and also the angle between the v-scale and the w-scale. The scaling factor m_3 corresponds to the w-scale. The zeros of the scales must meet at the vertex. If the angle A is chosen to be 60°, then 2 cos A = 1 and the three scaling factors are identical, as is the case in this figure.

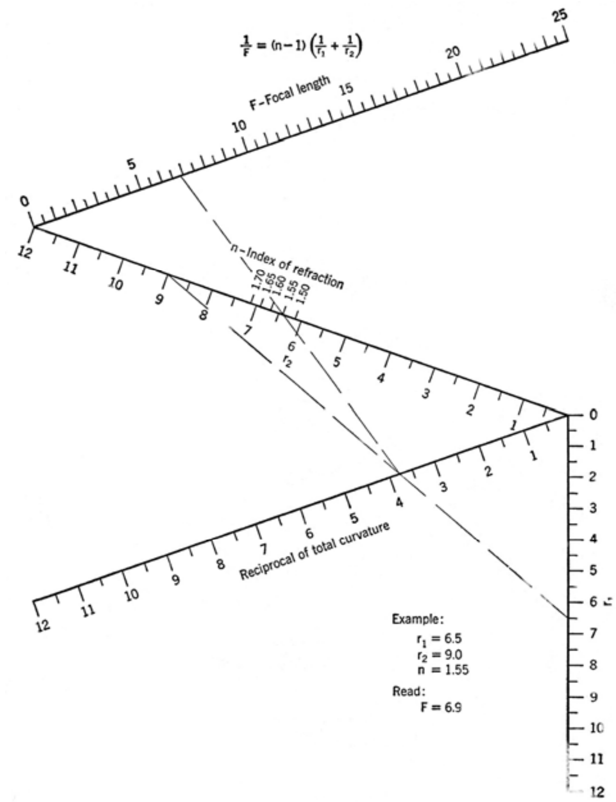


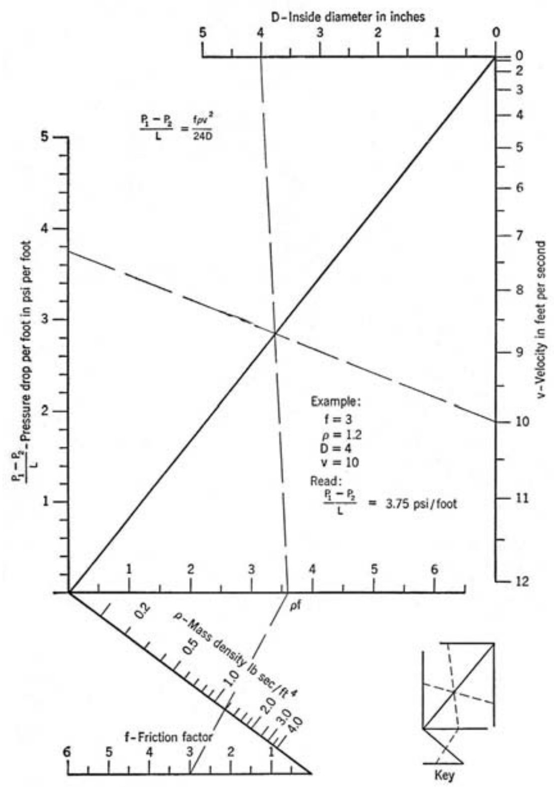
To solve the 4-variable equation $1/f_1(u) + 1/f_2(v) + 1/f_4(t) = 1/f_3(w)$, the equation is first re-arranged as $1/f_1(u) + 1/f_2(v) = 1/f_3(w) - 1/f_4(t)$. Then the two halves are set equal to an intermediate value f(q). A compound concurrent chart is then created in a similar way to other compound charts as shown in this figure (here A is chosen to be less than 60°).

4-Variable Charts

A 4-variable equation with one unknown can be represented as a combination of two separate charts of any type. The first step is to break the equation into two parts in three variables that are equal to one another. For $f_1(u) + f_2(v) + f_3(w) = f_4(t)$ and t unknown, we can re-arrange the equation into $f_1(u) + f_2(v) = f_4(t) - f_3(w)$ and create a new variable k to equal this sum. Then a *blank* scale for k is created such that a parallel scale nomogram for $f_1(u) + f_2(v) = k$ marks a **pivot point** on the k-scale, then a second straightedge alignment from this point is used for a parallel-scale nomogram for $f_4(t) - f_3(w) = k$ to find $f_4(t)$. The scaling for u, v and w and the position chosen for the k-scale can be optimized to minimize errors at the pivot point for small errors in the straightedge alignment. The figure on the right shows a compound parallel scale nomogram. Below are examples from Levens of compound nomograms of Z charts and concurrent and proportional charts. A key often provides instructions on the use of a compound nomogram as shown in the second figure. Of course, this concept can be extended to equations with additional variables, where color coding would be helpful.







Curved Scale Charts

It is possible to geometrically derive relationships for nomograms that have one or more curved scales, but the design of these more complicated nomograms is so much easier using determinants. Designing nomograms with determinants is the subject of the second part of this essay.

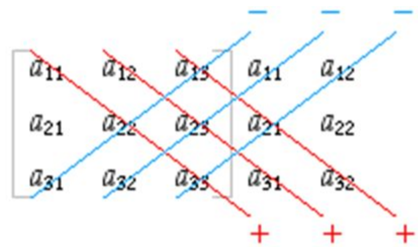
Designing with Determinants

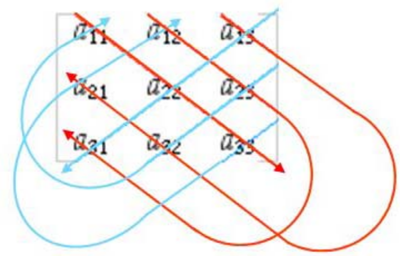
I've never had the slightest interest in matrices until I started researching this topic of nomography. I took a college class in linear algebra and found it tedious. But now I find that a brief knowledge of determinants offers a powerful way of designing very elegant and sophisticated nomograms. A few basics of determinants are presented here that require no previous knowledge of them. Then we will see how determinants can be manipulated to create extraordinary nomograms.

A **matrix** consists of functions or values arranged in rows and columns, as shown within the brackets in the figure on the right. The subscript pair refers to the row and column of a matrix element. A **determinant** represents a particular operation on a matrix, and it is denoted by vertical bars on the sides of the matrix. The determinant this 3x3 matrix is given by

$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$

But there are visual ways of deriving it. In the first figure the first two columns of the determinant are repeated to the right of the original, and then the products of all terms on diagonals from upper left to lower right are added and the products of all terms on diagonals from upper right to lower left are subtracted. A convenient mental shortcut is to find these diagonal products by "wrapping around" to get the three components of each term. Here the first product we add is the main diagonal $a_{11}a_{22}a_{33}$, then the second is $a_{12}a_{23}a_{31}$ where we follow a curve around after we pick up a_{12} and a_{23} to pick up the a_{31}, then $a_{13}a_{32}a_{21}$ by starting at a_{13} and wrapping around to pick up the a_{32} and a_{21}. We do the same thing right-to-left for the subtracted terms. This is much easier to visualize than to describe. Determinants of larger matrices are not considered here.





There are just a few rules about manipulating determinants that we need to know:

1. If all the values in a row or column are multiplied by a number, the determinant value is multiplied by that number. Note that here we will always work with a determinant of 0, so we can multiply any row or column by any number without affecting the determinant.

2. The sign of a determinant is changed when two adjacent rows or columns are interchanged.

3. The determinant value is unchanged if every value in any row (or column) is multiplied by a number and added to the corresponding value in another row (or column).

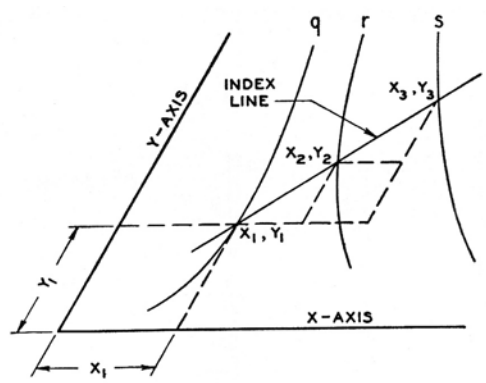
That's it. Now consider the general diagram to the right from Hoelscher showing three curvilinear scales and an isopleth. Similar triangle relations give

$$(y_3-y_1)/(x_3-x_1) = (y_2-y_1)/(x_2-x_1) = (y_3-y_2)/(x_3-x_2)$$

The first two parts of this can be rewritten as a cross product:

$$(x_3-x_2) / (y_2-y_1) = (x_2-x_1) / (y_3-y_2)$$

and when this is expanded it is equal to the determinant equation



$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

We get this result regardless of which pair we choose to use in the cross product. The x and y elements can be interpreted as the x and y values of $f_1(u)$, $f_2(v)$ and $f_3(w)$ if we don't mix variables between rows (the first row should only involve u, etc.) and if the determinant equation is equivalent to the original equation. This is the *standard nomographic form*. Here y is not expressed in terms of x as we normally have when we plot points at (x,y) coordinates, but rather x and y are expressed in terms of a third variable, that is, x_1 and y_1 are expressed in terms of a function of the variable u, x_2 and y_2 are expressed in terms of a function of the variable v, and x_3 and y_3 are expressed in terms of a function of the variable w. These are called *parametric equations*. One way to plot them is to algebraically eliminate the third variable between x and y to find a formula for y in terms of x. Another way is to simply take values of the third variable over some range, calculate x and y for each value, and plot the points (x,y)---a more likely scenario when we have computing devices.

Let's consider the equation $\mathbf{(u + 0.64)^{0.58}(0.74v) = w}$ that we used earlier to create a parallel scale nomogram. We converted this with logarithms to 0.58 log (u + 0.64) + log v = log w + log (0.74), or

$$0.58 \log (u + 0.64) + \log v - [\log w + \log (0.74)] = 0$$

We could have grouped log (0.74) with any term, but we'll stay consistent with our earlier grouping. This is an equation of the general form $\mathbf{f_1(u) + f_2(v) - f_3(w) = 0}$, so let's find a determinant that produces this form. We want each row to contain only functions of one variable of u, v or w, so we'll start with $f_1(u)$ in the upper left corner and 1's along its diagonal so that the first term of the determinant will be $f_1(u)$.

$$\begin{vmatrix} f_1(u) & ? & ? \\ ? & 1 & ? \\ ? & ? & 1 \end{vmatrix} = 0$$

Now let's set the lower left corner to $f_3(u)$ and the upper right corner to 1 so the last term of the determinant will be $-f_3(w)$:

$$\begin{vmatrix} f_1(u) & ? & 1 \\ ? & 1 & ? \\ f_3(w) & ? & 1 \end{vmatrix} = 0$$

Now we can place $f_2(v)$ in the middle row somewhere and fill in the rest of the determinant so the terms involving these elements end up simply as $f_2(v)$. Here are a few possibilities that work:

$$\begin{vmatrix} f_1(u) & 0 & 1 \\ f_2(v) & 1 & 0 \\ f_3(w) & 1 & 1 \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} f_1(u) & -1 & 1 \\ f_2(v) & 1 & 0 \\ f_3(w) & 0 & 1 \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} f_1(u) & -1 & 0 \\ f_2(v) & 1 & 1 \\ f_3(w) & 0 & 1 \end{vmatrix} = 0$$

But the second determinant is simply the first one after the third column is subtracted from the second column and the third determinant is simply the second one after the third column is added to the second column. These are operations that will not change our determinant equation as described in our earlier list, so they are all equivalent.

Now we want the flexibility to scale our $f_1(u)$ and $f_2(v)$ by m_1 and m_2 calculated for parallel scale charts as the desired height of the nomogram divided by the ranges of the functions. If we take the first determinant of the three possible ones shown, notice that we can introduce the scaling values without changing the determinant equation if we write it as

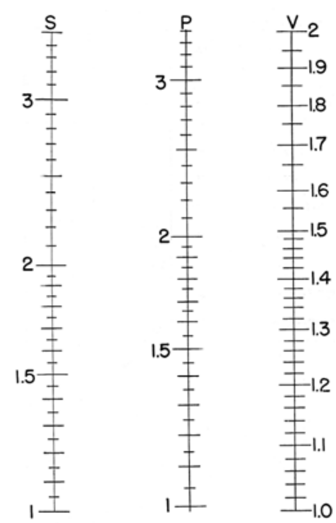
$$\begin{vmatrix} m_1 f_1(u) & 0 & 1 \\ m_2 f_2(v) & 1 & 0 \\ f_3(w) & 1/m_2 & 1/m_1 \end{vmatrix} = 0$$

So now we have to convert this to the standard nomographic form having all ones in the last column (and continuing to isolate variables to unique rows). First we add the second row to the third row, where $1/m_1 + 1/m_2 = (m_1+m_2)/m_1m_2$:

$$\begin{vmatrix} m_1 f_1(u) & 0 & 1 \\ m_2 f_2(v) & 1 & 1 \\ f_3(w) & 1/m_2 & \dfrac{m_1 + m_2}{m_1 m_2} \end{vmatrix} = 0$$

Then we multiply the bottom row by $m_1m_2/(m_1+m_2)$ and swap the first two columns so that the y column (the middle column) contains the functions:

$$\begin{vmatrix} 0 & m_1 f_1(u) & 1 \\ 1 & m_2 f_2(v) & 1 \\ \dfrac{m_1}{m_1 + m_2} & \dfrac{m_1 m_2}{m_1 + m_2} f_3(w) & 1 \end{vmatrix} = 0$$



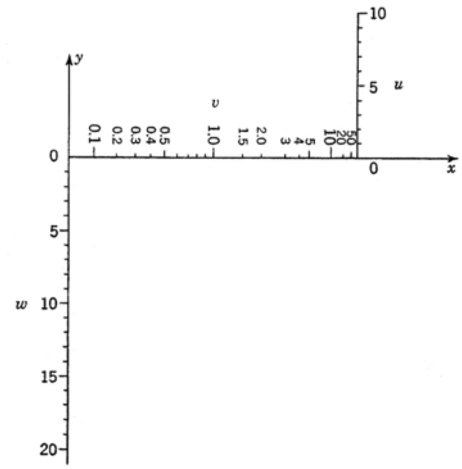
$(S+0.64)^{.58}(0.74V)=P$

and we have the determinant in standard nomographic form. The first column represents x values and the second column represents y values of the functions. The scaling factors of m_1 and m_2 result in a scaling factor m_3 for the w-scale of $m_1m_2/(m_1+m_2)$ as we found earlier from our geometric derivation. We had calculated $m_1 = 25.72$ and $m_2=19.93$ before, giving $m_3=11.23$. This determinant also shows that we place the u-scale vertically at x=0 and the y-scale vertically at x=1, with the w-scale at $x= m_1/(m_1+m_2) = 0.5634$, but in fact we can multiply the first column by 3 to get a scale of 3 inches, and in this case the w-scale lies vertically at x=1.69 inches, and so we end up with *exactly* the same nomograph we found earlier using geometric methods.

This was a bit of work, but we have found a universal standard nomographic form for the equation $\mathbf{f_1(u) + f_2(v) – f_3(w) = 0}$ including scaling factors.

Let's derive a Z chart for division using determinants. For **v** = **w/u**, we rearrange the equation so the right-hand term is 0, or **uv** – **w** = 0. One possible determinant we can construct is

$$\begin{vmatrix} u & 1 & 0 \\ 0 & v & 1 \\ -w & 0 & 1 \end{vmatrix} = 0$$



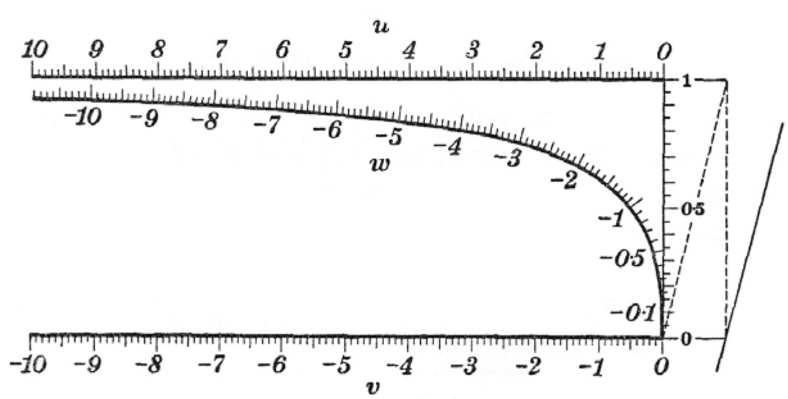
which graphs to the nomogram on the right, a Z chart with a perpendicular middle line. A different determinant would result in a Z chart of the more familiar angled middle line. An interesting aspect of such a chart is that the u-scale and v-scale have different scaling factors despite the fact that they can be interchanged in the equation.

There is a definite knack to all of this, and at this point I'd like to recommend the webpages on nomography by Winchell D. Chung, Jr. at this site. His webpages are quite interesting to read---there are quite a few examples of nomograms, and the determinant approach is used throughout. In particular, he provides other examples of expressing an equation into determinant form here. He also gives a few examples of converting the determinant to the standard nomographic determinant form here, where examples 2 and 3 are from Hoelscher. Most importantly, for equations of several standard formats Chung also reproduces tables that map these equations *directly* to standard nomographic determinant forms here.

Determinants are most useful when one or more of the u, v and w scales is curved. The quadratic equation $\mathbf{w^2 + uw + v = 0}$ can be represented as the first equation below, and dividing the last row by w-1 we immediately arrive at the standard nomographic form shown in the second equation:

$$\begin{vmatrix} -u & 1 & 1 \\ v & 0 & 1 \\ w^2 & w & w-1 \end{vmatrix} = 0 \qquad \begin{vmatrix} -u & 1 & 1 \\ v & 0 & 1 \\ \dfrac{w^2}{w-1} & \dfrac{w}{w-1} & 1 \end{vmatrix} = 0$$

The u-scale runs linearly in the negative direction along the line y=1. The v-scale runs linearly in the positive direction with the same scale along the line y=0. The x and y values for the curve for w can be plotted for specific values of w (a parametric equation), or w can be eliminated to express the curve in x and y as



x/y = w

y = (x/y) / [(x/y) – 1] = x / (x – y)

resulting in the figure at the right in which the positive root w_1 of the quadratic equation can be found on the curved scale (the other root is found as $u - w_1$).

Hoelscher presents the equation for the projectile trajectory $\mathbf{Y = X \tan A - gX^2 / (2V_0^2\cos^2 A)}$ where A is the initial angle, V_0 is the initial velocity, and g is the acceleration due to gravity. There are four variables X, Y, V_0 and A, so nomographic curves are plotted for different values of A. For an angle of 45º, the equation reduces to

$$Y = X - 0.0322X2 / V_0^2$$

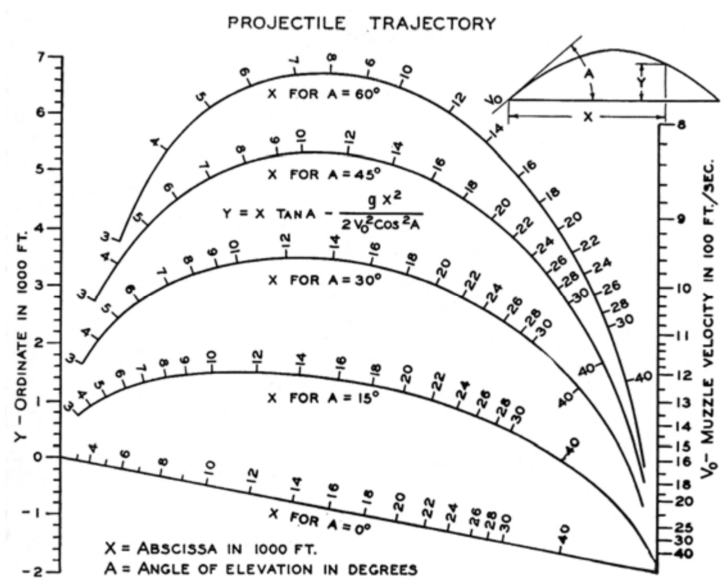
One determinant for this is shown in the first equation below, which can be manipulated into the standard nomographic form shown in the second equation:

$$\begin{vmatrix} 0.0322/V_0^2 & 0 & 1 \\ Y & X & 0 \\ 1 & 1 & X \end{vmatrix} = 0 \qquad \begin{vmatrix} 0.0322/V_0^2 & 0 & 1 \\ Y & 1 & 1 \\ \dfrac{X}{X^2+1} & \dfrac{1}{X^2+1} & 1 \end{vmatrix} = 0$$

Hoelscher assumes -2000<Y<7000 ft and 800<V_0<4000 fps and a chart of 5 inches square, so after some more manipulations (including the swapping of the first two columns) we arrive at the final form:

$$\begin{vmatrix} 0 & 0.0322/V_0^2 & 1 \\ 5 & 0.5Y & 1 \\ \dfrac{1000}{X^2+200} & \dfrac{100X}{X^2+200} & 1 \end{vmatrix} = 0$$

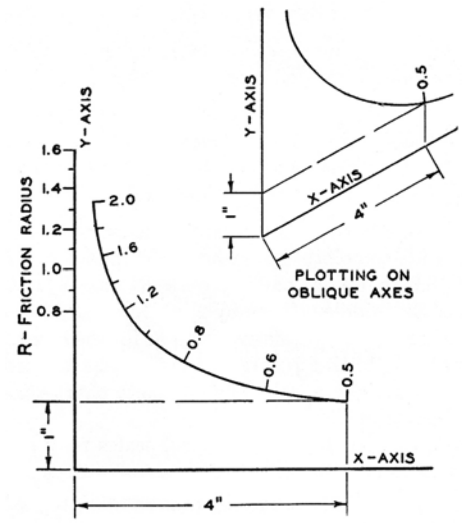
This is shown as the curve for A=45º along with curves for other angles in this figure (a *grid nomogram* such as this can be used to handle an equation with more than 3 variables).

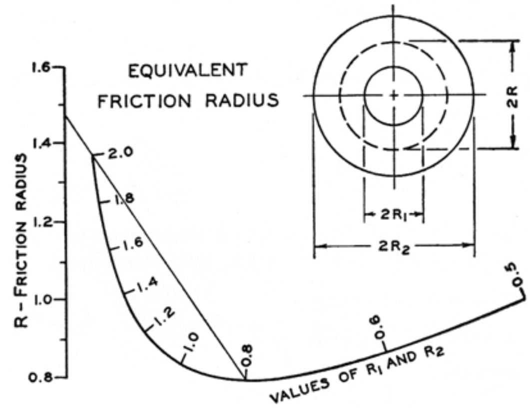


It's possible to have two or three scale curves depending on how the determinant works out, and it is possible to have two or all three curves overlap exactly. The equation for the equivalent radius of the friction moment arm for a hollow cylindrical thrust bearing is $\mathbf{R = 2/3\,[R_1^3 - R_2^3] / [R_1^2 - R_2^2]}$ or $3RR_1^2/2 - 3RR_2^2/2 - R_1^3 + R_2^3 = 0$, yielding the determinant

$$\begin{vmatrix} \dfrac{1}{R_2^2} & 2R_2 & 1 \\ 0 & 3R & 1 \\ \dfrac{1}{R_1^2} & 2R_1 & 1 \end{vmatrix} = 0$$

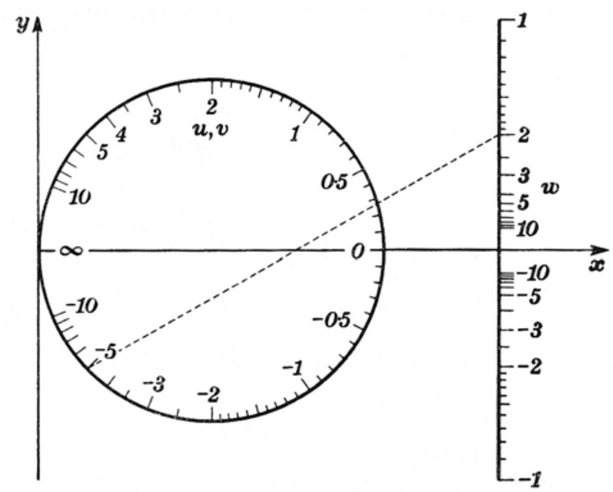
Here the R_1 and R_2 scales lie exactly on the same curve. They could have separate tick marks on this curve if they had a different scale, but here they have the same scaling factor. The figures here show the plotted nomogram for perpendicular x and y axes, and also for oblique axes that expand the R-scale to the full height of the nomogram for greater accuracy.





Otto provides an interesting alternate determinant for the equation $\mathbf{f_1(u) + f_2(v) + f_3(w) = f_1(u)\, f_2(v)\, f_3(w)}$.

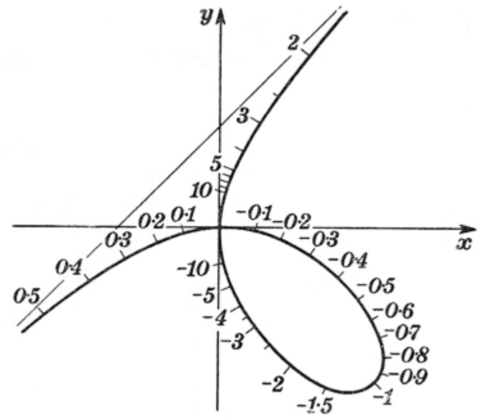
$$\begin{vmatrix} \frac{2}{f_1(u)^2+4} & \frac{f_1(u)}{f_1(u)^2+4} & 1 \\ \frac{2}{f_2(v)^2+4} & \frac{f_2(v)}{f_2(v)^2+4} & 1 \\ \frac{2}{3} & \frac{1}{3 f_3(w)} & 1 \end{vmatrix} = 0$$



The nomogram for the particular equation of this type **u + v + w = uvw** is shown in this figure, where again two of the three scales overlap. Eliminating $f_1(u)$ from the x and y elements in the first row we find that u lies along the circle given by $(x - ¼)^2 + y^2 = 1/(4^2)$, and this is correspondingly true for v as well (although in general we have to separately calculate x and y tick marks based on $f_1(u)$ or $f_2(v)$ values).

Finally, Otto describes a very interesting determinant that can be created for the equation $\mathbf{f_1(u)\, f_2(v)\, f_3(w) = 1}$.

$$\begin{vmatrix} \dfrac{-f_1(u)}{1-f_1(u)^2} & \dfrac{-f_1(u)}{1-f_1(u)^3} & 1 \\ \dfrac{-f_2(v)}{1-f_2(v)^2} & \dfrac{-f_2(v)}{1-f_2(v)^3} & 1 \\ \dfrac{-f_3(w)}{1-f_3(w)^2} & \dfrac{-f_3(w)}{1-f_3(w)^3} & 1 \end{vmatrix} = 0$$



For **uvw=1**, all three scales coincide and have the same scaling factor, and it turns out that the equation for this curve is $x^3 + y^3 - xy = 0$ (called the *folium of Descartes*). This nomogram is shown as the curled figure to the right.

Transformations

In addition to providing sophisticated nomograms, the use of determinants offers one other huge advantage. Often the scaling factors of variables have to be manipulated to get a nomogram that uses all the available area and yet stretches portions of the curves that are most in need of accuracy; alternatively, there may be a need to bring distant points (even at infinity) into a compact nomogram. This can be done experimentally (Chung suggests using a spreadsheet) or by projecting the nomogram in any manner that maps points into points and lines into lines. This can be tedious using geometric formulas, but it can be done by multiplying the determinants by standard translation and rotation matrices. Let's look at the types of transformations that can be used for a nomogram, based on the extended presentation in Epstein.

Translation

We can translate the nomogram laterally, which is equivalent to translating the x-y axes to new x'-y' axes. We can add *c* to all determinant elements in the x column and *d* to all determinant elements in the y column to shift the axes left by *c* and down by *d* (or in other words shift the nomogram right by *c* and up by *d*).

$$x_n' = x_n + c$$

$$y_n' = y_n + d$$

Rotation

We can rotate the nomogram about the origin of the axes by an angle θ (positive for counter-clockwise rotation) by replacing each determinant element x_n in the x column and the each determinant element y_n in the y column with

$$x_n' = x_n \sin\theta + y_n \cos\theta$$

$$y_n' = x_n \cos\theta - y_n \sin\theta$$

Stretch

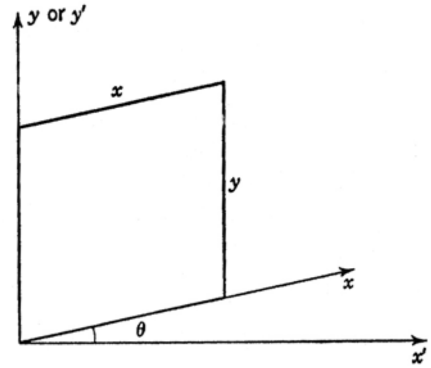
We can stretch a nomogram in the x direction by multiplying each determinant element in the x column by a constant, and likewise for the y direction.

$$x_n' = c x_n$$

$$y_n' = d y_n$$

Shear

Shear is a slewing of perpendicular axes to oblique axes or vice-versa. This is perhaps best understood by referring to this figure showing a shear from one set of axes x-y to another set x'-y' in which the x' axis is canted at an angle θ to the x axis but the y' axis aligns with the y axis. For this case,



$$x_n' = x_n \cos\theta$$

$$y_n' = y_n + x_n \sin\theta$$

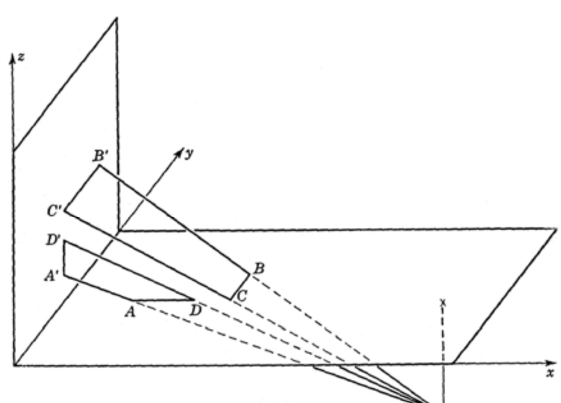
Shear can be used to convert a traditional Z chart with a slanting middle line to one with a perpendicular middle line and vice-versa. It could have been used to convert the determinant for the earlier nomogram of equivalent friction radius to plot it relative to the oblique x'-y' axes rather than to the perpendicular x-y axes.

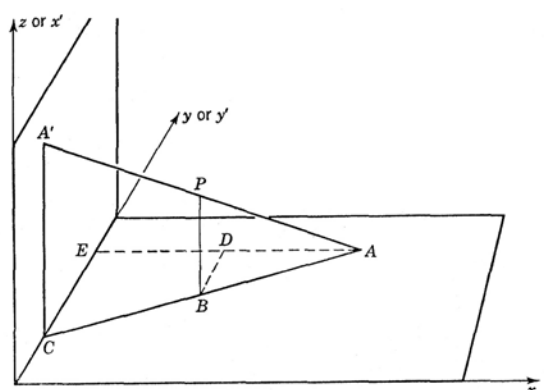
Projection

Referring to the first figure below, a projection uses a point P (called the *center of perspectivity*) to project rays through points of a nomogram in the x-y plane to map them onto the z-y plane (also called the x'-y' plane), foreshortening or magnifying lines in varying amounts in the x' and y' directions. It is also possible for P to lie above the x-y plane, where rays from points on the nomogram pass through P to the x'-y' plane as shown in the second figure. (This can be used to convert a nomogram in the shape of a trapezoid to a rectangular one, changing scale resolutions to maximize the available space, but we will see an easier method later.) In either case, for a P location (x_P, y_P, z_P) and a nomogram x-element x_n and y-element y_n,

$$x_n' = z_P\, x_n / (x_n - x_P)$$

$$y_n' = (y_P\, x_n - x_P\, y_n) / (x_n - x_P)$$





The line BD under P in the second figure never maps onto the x'-y' plane because it is parallel to that plane—it is a “straight line at infinity” and will become important in our example below.

A Transformation Example

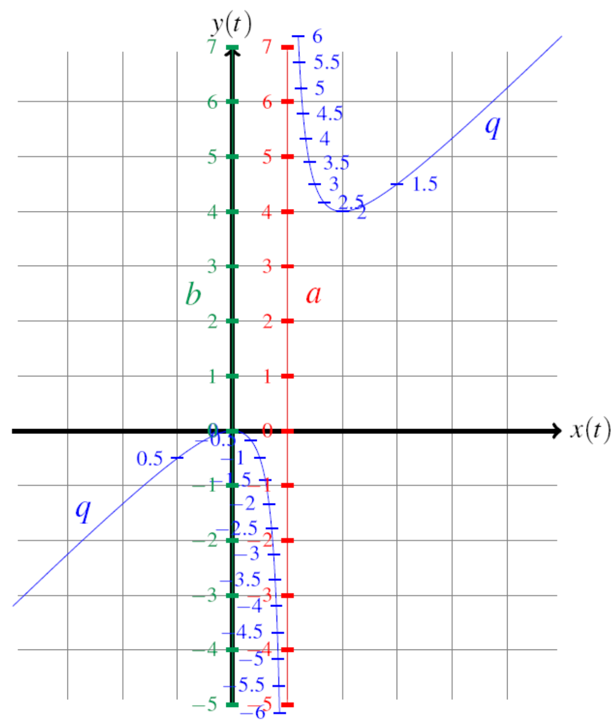
Epstein outlines a sequence of transformations to convert a nomogram for the equation $\mathbf{q^2 - aq + b = 0}$ to a more convenient circular form. Unfortunately, he provides only the final nomogram, so I have taken up his challenge and traced through the details of each step while creating my own intermediate nomograms. These nomograms were created using the freely-available LaTeX typesetting engine and the free vector drawing package TiKz, which supports plotting parametric functions and has enough flexibility to draw and nicely label tick marks on the curves. Excel and MATLAB, for example, can plot parametric functions but do not appear to support labeled tick marks on the curves corresponding to the parametric variable. Chung uses Python code to create his nomograms (described here) and there is a Python program to plot nomograms here that I have not evaluated. An online tool to create custom, interactive parallel scale nomograms only can be found here. I find that the LaTeX code is quite simple and very flexible and is especially convenient for those of us who already use LaTeX to create technical articles. My LaTeX code that created the nomograms below can be found here.

A determinant representing the equation $\mathbf{q^2 - aq + b = 0}$ can be constructed (and verified it by multiplying it out) as

$$\begin{vmatrix} q/(q-1) & q^2/(q-1) & 1 \\ 1 & a & 1 \\ 0 & b & 1 \end{vmatrix} = 0$$

(Note: In all determinants in this essay the x elements are in the first column and the y elements in the second column, which follows most presentations but is reversed from Epstein’s.)

This nomogram is plotted in the figure to the right (the x-y axes and grid would be deleted from the final nomogram). The q-scale is found in practice by taking a range of q values and calculating x = q / (q+1) and y = q^2 / (q-1) as a coordinates to plot, but if we eliminate q between the two parametric equations we arrive at $x^2 - xy + y = 0$, demonstrating that the q-scale is in fact a hyperbola. A straightedge placed across any two values of a and b will intersect the q-scale in two points if there are two real solutions, one point if there is a double real root, and no points if there are no real roots.



However, the layout of the q-scale is problematic, as the two halves stretch toward infinity very quickly and it is not possible to accurately locate q points for isopleths near the asymptotes of the hyperbola. So we will transform this nomogram into one in which the q-scale is finite. The labels for the tick marks will not be displayed on the following plots, but the tick mark spacing and colors will provide a guide for how the curves are re-mapped.

First we will rotate this nomogram clockwise by 45° (or θ = -45°) and stretch it in both dimensions by $2^{1/2}$ for a reason that will become apparent in the next transformation. Since cos -45° = $2^{-1/2}$ and sin 45° = $-2^{-1/2}$, the earlier rotation formulas after the stretch become

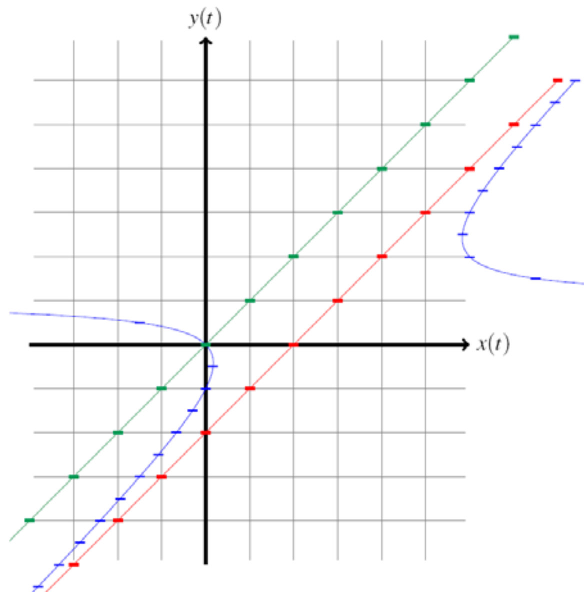
$$x_n' = x_n + y_n$$

$$y_n' = -x_n + y_n$$

Performing this substitution for the x element and y element of each row of the determinant, we arrive at

$$\begin{vmatrix} (q^2+q)/(q-1) & q & 1 \\ a+1 & a-1 & 1 \\ b & b & 1 \end{vmatrix} = 0$$

which is plotted on the right.



We rotated the nomogram because we wanted a vertical line (say, x=1) that does not intersect the hyperbola. A projection transformation can convert a scale with two branches like this hyperbola into a single connected scale (an ellipse) if a straight line separating the two branches is projected to infinity, that is, if the line is parallel to the y-z axis (which x=1 is) and the projection point P is located directly above or below it in its z value (as the line BD in the earlier projection figure). Choosing P = (1,-1,1), the earlier projection formulas become

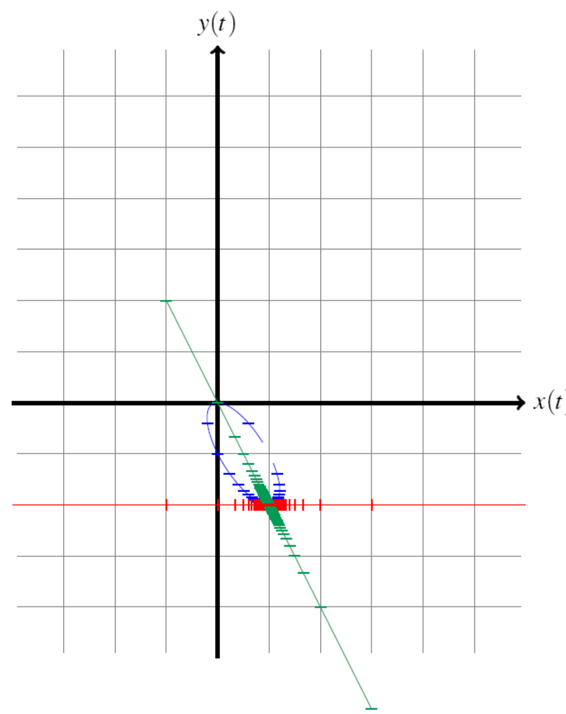
$$x_n' = x_n / (x_n - 1)$$

$$y_n' = (-x_n - y_n) / (x_n - 1)$$

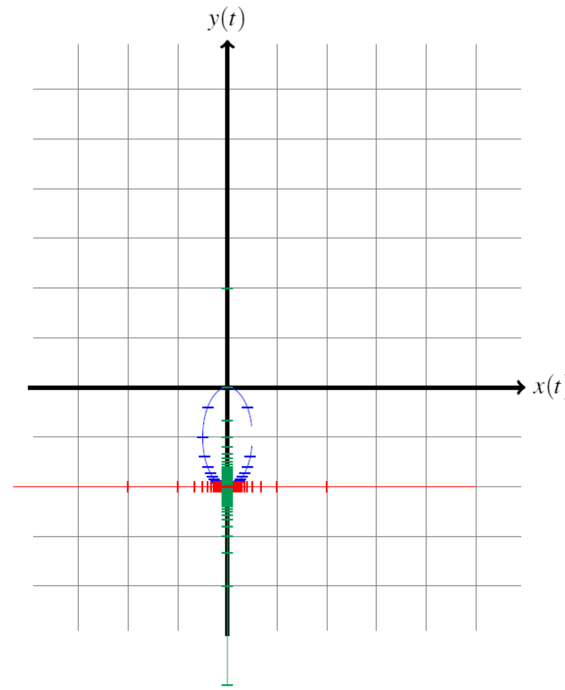
and the determinant becomes

$$\begin{vmatrix} (q^2+q)/(q^2+1) & -2q^2/(q^2+1) & 1 \\ (a+1)/a & -2 & 1 \\ b/(b-1) & -2b/(b-1) & 1 \end{vmatrix} = 0$$

and the ellipse magically appears in the plot on the right.



Now let's shear the nomogram so the green b-scale lies on the y-axis while keeping the red a-scale parallel to the x-axis. The shear formulas are slightly different as we are shearing to the y-axis, and for a b-scale slope of –½ they reduce to:



$$x_n' = x_n + y_n / 2$$

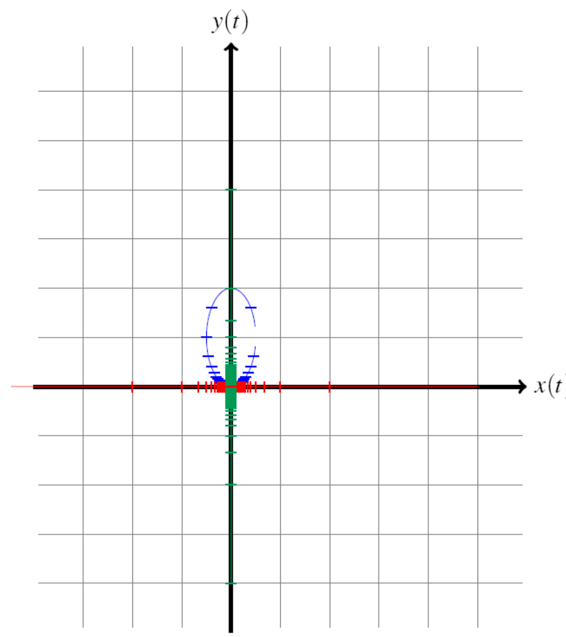
$$y_n' = y_n$$

and the determinant becomes

$$\begin{vmatrix} q/(q^2+1) & -2q^2/(q^2+1) & 1 \\ 1/a & -2 & 1 \\ 0 & -2b/(b-1) & 1 \end{vmatrix} = 0$$

which is plotted on the right.

Now we'll translate the nomogram upward by 2 to place the intersection point on the origin.



$$x_n' = x_n$$

$$y_n' = y_n + 2$$

$$\begin{vmatrix} q/(q^2+1) & 2/(q^2+1) & 1 \\ 1/a & 0 & 1 \\ 0 & -2/(b-1) & 1 \end{vmatrix} = 0$$

This is plotted on the right.

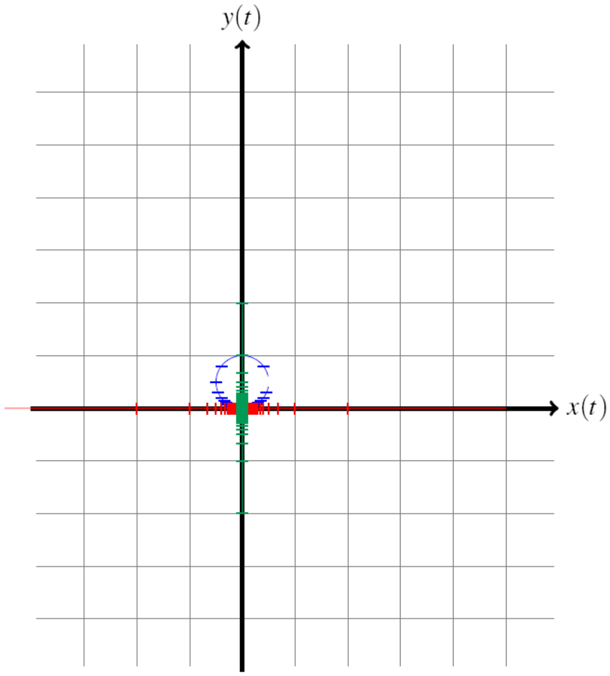
And finally we shrink the nomogram in the y direction by a factor of 2 to get a circular scale for q.

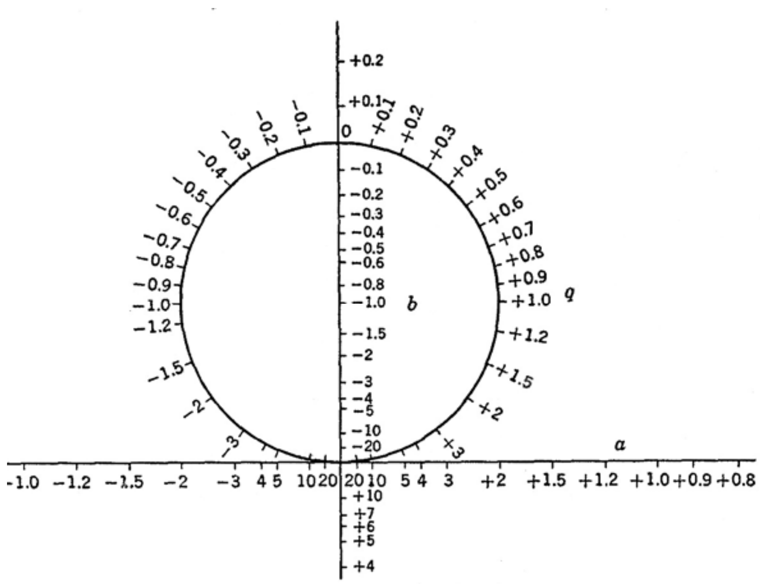
$$x_n' = x_n$$

$$y_n' = y_n / 2$$

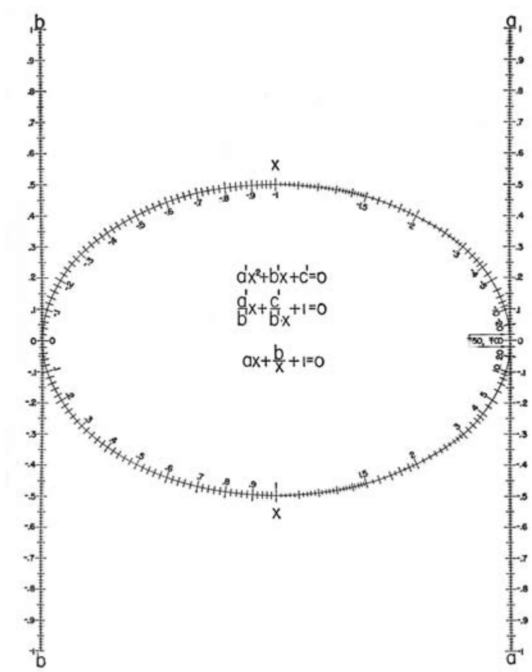
$$\begin{vmatrix} q/(q^2+1) & 1/(q^2+1) & 1 \\ 1/a & 0 & 1 \\ 0 & -1/(b-1) & 1 \end{vmatrix} = 0$$

The figure on the left below is the plot of my final determinant, and by changing the scales of the axes we arrive at Epstein's figure on the right.





The entire range of q from -∞ to +∞ is now represented in a finite area, and certainly the range less than 1.5, which veered to infinity in our original nomogram, is nicely accessible. The larger numbers are not as accessible, but the ranges can be skewed to spread out any range by multiplying the original equation by a constant. We could have stopped at any of the nomograms containing an ellipse, but it is easier to draft the circle. An elliptical nomogram for the quadratic equation $ax^2 + bx + c = 0$ is shown here for comparison.



It's interesting to play around with a straightedge on the circular nomogram we derived above to see that it works. In particular, an isopleth through an a-value and b-value will just touch the q-circle if the discriminant from the quadratic formula is 0 (for the equation $Ax^2 + Bx + C = 0$, the discriminant is the value B^2-4AC whose square root is taken in the quadratic formula, or a^2-4b here). When the discriminant is less than zero the isopleth misses the q-circle, denoting no real roots, and when it is greater than zero it crosses two real roots on the q-circle.

And in fact if you eliminate the a-scale, then the b-scale represents the product of two numbers on the q-circle. This is because if we have two solutions q_1 and q_2 of $\mathbf{q^2 - aq + b = 0}$, then the equation can be written as $\mathbf{(q - q_1)(q - q_2) = 0}$. Multiplying this out and equating terms to the original equation, we find that $b = q_1 q_2$ and $a = q_1+q_2$. The geometric layout of this nomogram is really the traditional meaning of a circular nomogram (two scales on the circle, one on a nearby line), and it turns out that any three-line parallel scale nomogram can be transformed into a circular nomogram. Generally the two scales on the circle do not have the same values, so tick marks on both sides of the circumference are needed.

The transformations we have discussed can also be represented as matrices. Transformations are performed by matrix multiplication of the transformation matrix and the nomogram determinant. Two or more transformations can be combined by multiplying their transformation matrices. It often happens after such a matrix multiplication that the nomogram determinant needs to be manipulated again into the standard nomographic form. For example, the transformation matrices for rotation and projection are

$$\begin{vmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} -x_P & 0 & 0 \\ y_P & z_P & 0 \\ 0 & 0 & -x_P \end{vmatrix}$$

It is possible to use matrix multiplication to map a trapezoidal shape (such as the boundaries of a nomogram that does not occupy a full rectangle) into a rectangular shape. This would increase the accuracy of the scales that can be expanded to fill the sheet of paper. Consider the following matrix multiplication:

$$\begin{vmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{vmatrix} \quad \text{x} \quad \begin{vmatrix} x_u & y_u & 1 \\ x_v & y_v & 1 \\ x_w & y_w & 1 \end{vmatrix}$$

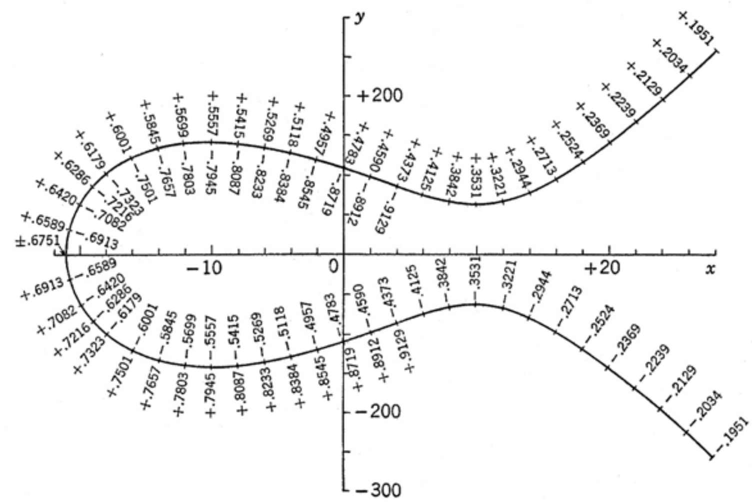
By the rules of matrix multiplication and some manipulation of the result, each y' and x' in the resulting matrix can be represented as

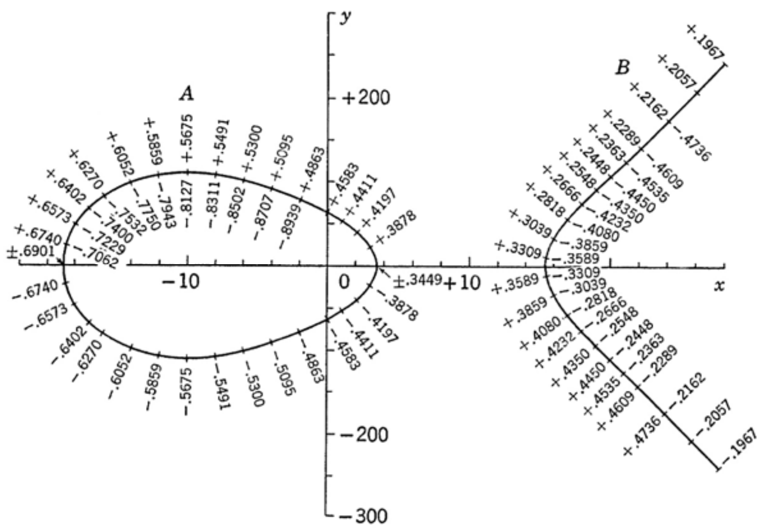
$$x' = (xk_{11} + yk_{21} + k_{31}) / (xk_{13} + yk_{23} + k_{33})$$

$$y' = (xk_{12} + yk_{22} + k_{32}) / (xk_{13} + yk_{23} + k_{33})$$

Now if we want to remap an area such that the points (x_1,y_1), (x_2,y_2), (x_3,y_3) and (x_4,y_4) map to, say, the rectangle (0,0), (0,a), (b,0) and (b,a), we insert the final and initial x's and y's into the formulas above, giving us eight equations in nine unknown k's. So we choose one k, solve for the other eight k's and multiply the original nomogram determinant by the k matrix and convert it back to standard nomographic form, then replot the nomogram---a fun way to spend an afternoon.

There are non-projective transformations as well that can be used to create nomograms in which all three scales are overlaid onto one curve (although the third will have different tick marks). This is highly mathematical and involves things called Weierstrass' Elliptic Functions, so Epstein is a resource if there is interest in the details. Epstein provides nomograms of this sort for the equation **u + v + w = 0** (which can be generalized to any equation of this form, including ones in logarithms). In the first figure below, the isopleth must cross two numbers on one scale and a third number on the other overlaid scale (such as u = +0.2524, v = +0.3842 and w= -0.6366). In the second figure the isopleth crosses two curves containing three scales. Ignore the numbers on the x-axis and y-axis---these relate to the function used to derive the nomograms. I'm showing these simply to demonstrate the advanced mathematics that was targeted at nomographic construction at one time.

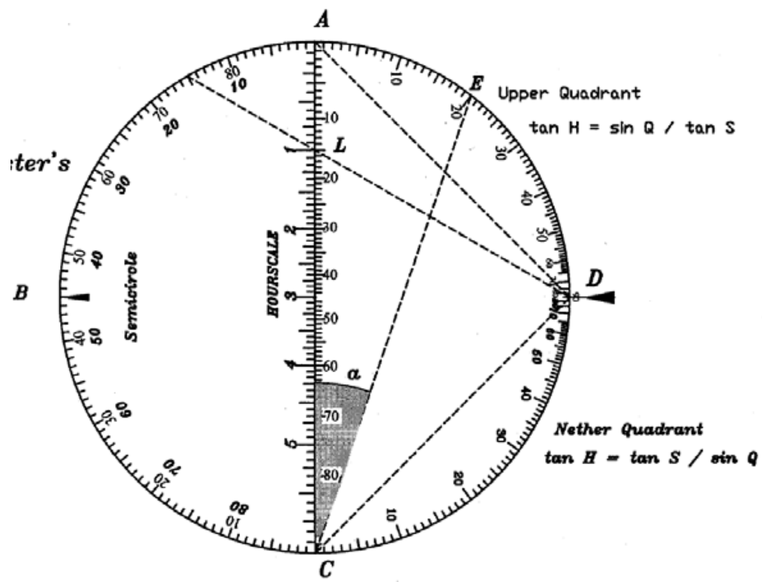


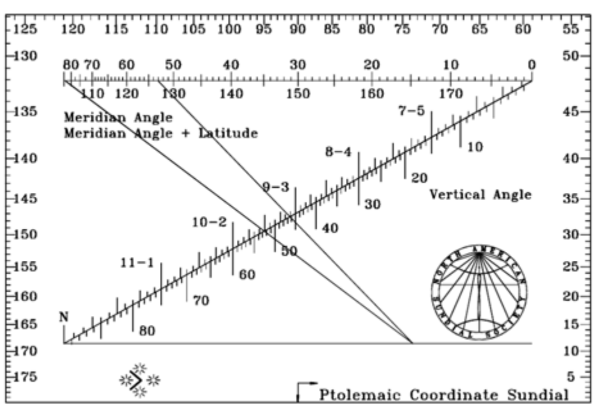


The Status of Nomograms

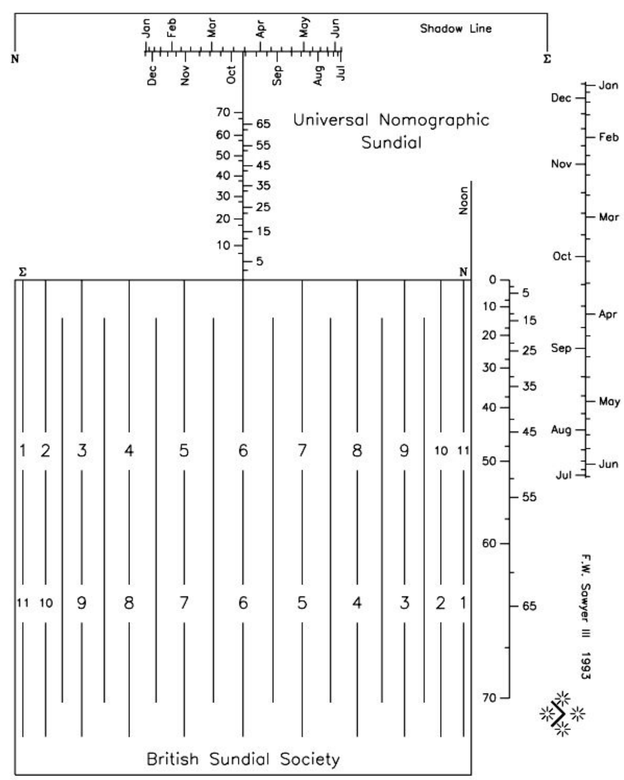
Today the use of nomograms seems scattered at best. Chung's interest grew from his desire to provide quick and easy calculations of hit strength, etc., in strategic wargaming. There are many simple nomograms that exist for doctors to quickly assess attributes and probabilities, such as here (or here if you can't draw a line), here, here, and here. A Body Mass Index (BMI) nomogram is common (such as here) and is derived in this PowerPoint presentation on nomograms. There are also some engineering nomograms found here and there online (such as here, here, here, here, here, and there).

I first heard of nomograms in the context of sundials. As Vinck and Sawyer describe (see references), Samuel Foster published a treatise in 1638 titled "The Art of Dialling" that contained a dialing scale for the construction of horizontal, vertical and inclining sundials as shown in the figure from Vinck to the right. Foster's construction scale is actually a circular nomogram, a tool discovered nearly 300 years earlier than its attributed discovery by J. Clark in 1905! Sawyer writes that Foster did write of the more general computing applications of his scale. Here an isopleth from one point on a perimeter scale through a point on the middle scale will cross their product on the other perimeter scale, and with suitable trigonometric scaling one can lay out hour lines on a wide variety of sundials. Certainly many sundial designs (nearly all) rely on graphical plots with the gnomon shadow or a weighted, hanging string serving as the isopleth. Card dials are particularly complicated because they map a 3-D geometry to a 2-D plane, as you can see on this webpage. Sawyer has designed a few dials that employ nomograms, two of which are shown below. Masse uses the same projection transformation method as we did to create a sundial in which the gnomon tip shadow during the day traces a circle rather than a hyperbola on the face of the dial. My interest in nomography (including the transformation techniques) and my efforts to plot nomograms using the LaTeX typesetting engine are partly due to my intention to create new sundial designs based on nomograms.

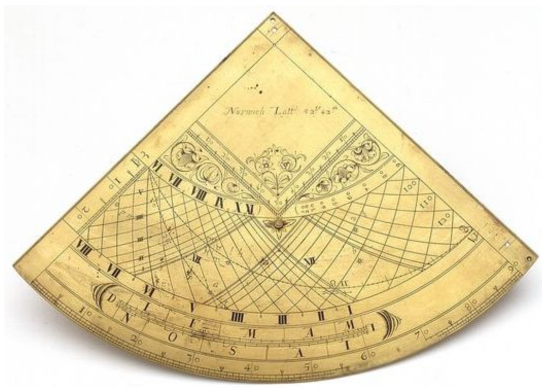


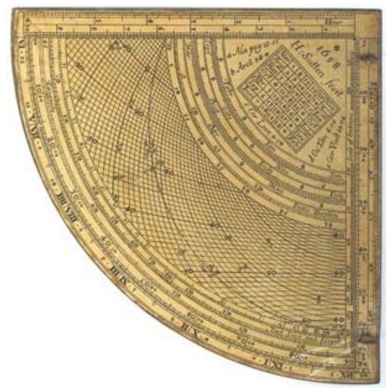


Latitude = 42 Meridian = 53.0 Vertical = 46.3 Time is 2:30



The use of analog graphic calculators is actually much older---the quite complicated grids and curves on old astrolabes and quadrants, as shown below, effectively serve as nomograms. There are also curves on the backs of most astrolabes to convert equal hours to unequal hours using the alidade (the sight on the back) as an isopleth, and the *qibla* diagram on the back of Islamic astrolabes provides the direction to Mecca for any hour of any day by using the alidade when the astrolabe is aligned correctly.





And I think there could be more applications today. I was at a picture framing store while I was writing this essay to get a matte board cut as a frame for some calligraphy created by my son. The pricing was based on three variables (the window height, the window length and the border width) and possibly the matte board type. Everyone who came up to get a price for a certain configuration or a variety of configurations had to wait while the clerk wrote down the three parameters, punched them into some formula on a calculator, and referred to a chart to find the corresponding price in quarter-dollars. I was thinking the whole time that having photocopies of a nomogram laying around would let customers use a straightedge (and there are a lot of those in a framing store!) to figure the pricing out and quickly optimize their parameters without having to wait in line, and it would certainly be faster for the clerk.

But nomograms have their own intrinsic charm. As a calculating aid a nomogram can solve very complicated formulas with amazing ease. And as a curiosity a nomogram provides a satisfying, hands-on application of interesting mathematics in an engaging, creative activity.

References

Douglass, Raymond D. and Adams, Douglas P. **Elements of Nomography**. New York: McGraw-Hill (1947). This book is as concerned about the drafting aspects of nomograms as with their (exclusively geometric) derivations. The printed forms it champions for calculating nomograms are more trouble than they are worth, in my opinion.

Epstein, L. Ivan. **Nomography**. New York: Interscience Publishers (1958). An advanced book that treats determinants throughout as well as projective transformations. It should not be the first book or two to read on nomography.

Hoelscher, Randolph P. *et al*. **Graphic Aids in Engineering Computation**. New York: McGraw-Hill (1952). The best all-around book on nomography of the ones listed here. Geometric derivations are given first and determinants introduced later in a very understandable presentation. It also contains a very interesting chapter on special slide rules created for solving particular engineering equations.

Johnson, Lee H. **Nomography and Empirical Equations**. New York: John Wiley and Sons (1952). A good, solid reference that is very readable. Uses geometric derivations throughout (no determinants) and treats all the common nomographic forms.

Levens, A.S. **Nomography**. New York: John Wiley and Sons (1937). A very readable book that is organized in a very convenient manner based on equation types. A very nice collection of nomogram examples of various types is presented in the Appendix. For nomograms created from geometric derivations only (and only the most sophisticated nomograms types really require solutions by determinants) this is an excellent reference book with many examples.

Masse, Yvon. *Central Projection Analemmatic Sundials*, The Compendium, Journal of the North American Sundial Society, Vol. 5, No. 1, Mar. 1998, pp.4-9.

Otto, Edward. **Nomography** (International Series of Monographs in Pure and Applied Mathematics).New York: Macmillan (1963). An advanced but (with some work) a readable book that uses determinants throughout to create nomograms as well as projective transformations to square them up or expand scales for greater accuracy. There is a long, interesting chapter on problems of theoretical nomography, such as determining what equation forms can be expressed as nomograms.

Sawyer, Fred. *Ptolemaic Coordinate Sundials*, The Compendium, Journal of the North American Sundial Society, Vol. 5, No. 3, Sep. 1998, pp.17-24. Describes three sundials incorporating nomograms.

Sawyer, Fred. *A Universal Nomographic Sundial*, Bulletin of the British Sundial Society, Oct. 1994. 94(3):14-17, also in **Sciatheric Notes**, a collection of sundial articles by Fred Sawyer from this publication. This collection and all the sundial-related articles referenced here can be found in the outstanding North American Sundial Society Repository CD orderable on the NASS site.

Vinck, Rene J. *Samuell Foster's Circle*, The Compendium, Journal of the North American Sundial Society, Vol. 8, No. 3, Sep. 2001, pp.7-10. Also see Fred Sawyer's *The Further Evolution of Samuel Foster's Dialing Scales* that follows this article in the same issue of the journal.