Lightning Calculators

Dead Reckonings: Lost Art in the Mathematical Sciences
http://www.myreckonings.com/wordpress
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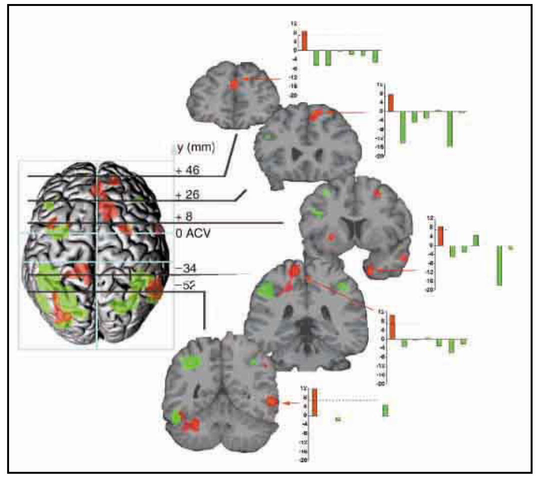
Individuals with preternatural abilities to calculate arithmetic results without pen, paper or other instruments, and to do so at astonishing speed, are the stuff of mathematical and psychological lore. These "lightning calculators" were sometimes of limited mental ability, sometimes illiterate but of average intelligence, and sometimes exceptionally bright, this despite the popular notion of the *idiot savant*. The techniques used by these people are not generally well known. In fact, despite claims by educators that acquiring a mental facility with arithmetic operations is essential to a student's mathematics education, I see little in the textbooks other than simple estimations based on rounding values, surely the most basic and least interesting mental task. The field of mental calculation may not be a lost art *per se*, but in this digital age it most certainly is a neglected one.

Part I of this essay attempts to take a fresh look at both historical and modern lightning calculators. Part II describes classic and modern methods of mental calculation. And finally, Part III demonstrates as a cautionary tale the shallow and deceptive nature of most media coverage of lightning calculators, an important consideration in analyzing reports on them.

The subject of lightning calculation has been an interest of mine for many years. Although I'm certainly not a lightning calculator, as a graduate teaching assistant in physics in the early 80's I enjoyed mentally calculating the results of problems to quite high accuracy while the students were working their calculators, and I would typically end the semester with a class on such methods. In 1988 I started gathering material for a book on methods of high-precision mental calculation of arithmetic as well as elementary functions such as logarithms, exponentials and trigonometric functions (*Dead Reckoning: Calculating Without Instruments*, 1993). A few years ago I started my main website MyReckonings.com mainly to devote a portion of it to notes and errata for the book, as well as serve as a repository for papers I've written on topics of mental calculation. The website area devoted to the book is http://www.myreckonings.com/Dead_Reckoning/Dead_Reckoning.htm and the page devoted to additional papers and is found at http://www.myreckonings.com/Dead_Reckoning/Online/Online_Material.htm, where some of the links in this essay are directed.

Part I: The Players



The history of lightning calculators, at least to 1983, is presented most comprehensively in Steven B. Smith's book, *The Great Mental Calculators: The Psychology, Methods and Lives of Calculating Prodigies Past and Present*. This is a fascinating read and an honest attempt to analyze the capabilities and methods of a number of these individuals. Smith notes that isolation (at least mental isolation), generally in children, is a condition favoring the development of this ability, and it's hard to argue with that. He describes a gamut of lightning calculators who run the spectrum of mental acuity.

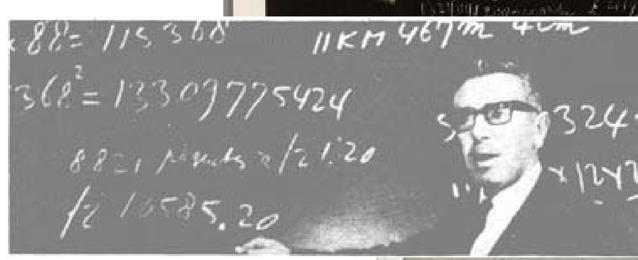
So let's see who we have in the book. Grouping people by their overall mental acuity is a dangerous sport prone to misinterpretation and error (for example, it is impossible for me to classify Thomas Fuller (1710-1790) because he was a victim of the slave trade in America). As a rough interpretation by me from Smith's book, among those that seem to be of low intelligence (which represents a large range) are

- Jedidiah Buxton (1702-1792)
- Henri Mondeux (1826-1861)
- Jacques Inaudi (1867-1950)

Among those who seem of average intelligence are

- Zerah Colburn (1804-1839)
- Johann Martin Zacharias Dase (1824-1861)
- Pericles Diamandi (1868-)
- Arthur Griffith (1880-1911)
- Salo Finkelstein (1896/7-?)
- Maurice Dagbert (1913-?)

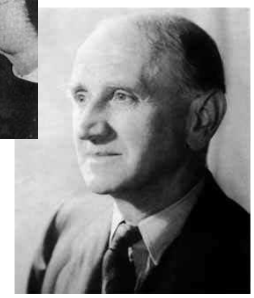
Those who seem to have exceptional intelligence include



- George Parker Bidder (1806-1878)
- Truman Henry Safford (1836-1901)
- Frank D. Mitchell
- Gottfried Ruckle (1879-1929)
- Wim Klein (1912-1986)
- Hans Eberstark (1929-)
- Shyam Marathe (1931-)
- Shakuntala Devi (1932-)
- Arthur Benjamin (1961-)

And those with ability in this area who left a permanent mark on mathematics and science certainly include

- Alexander Craig Aitken (1895-1967)
- John Wallis (1616-1703)
- Andrè Ampere (1775-1836)
- Leonhard Euler (1707-1783)
- Karl Frederich Gauss (1777-1855)
- John von Neumann (1903-1957)



So in fact we see a predominance of ability in those with higher mental acuity, and it turns out that as we proceed to later names here we find that these abilities remained or were developed in adulthood. Savants are certainly more fascinating because of their lack of ability in other areas, but the talents of those without disabilities is certainly in contrast to the popular conception of lightning calculators.

Pictures from top: Jedidiah Buxton, Jacques Inaudi, Wim Klein, Wim Klein, T. H. Safford., Maurice Dagbert, A.C Aitken

Watch a child who is doing math homework—when they are calculating the answer they generally get quite physically agitated, tapping a pencil, shaking or hitting their heads, standing up and sitting down, talking, etc. It’s quite striking when you are looking for it, a strange association between mathematical reasoning and motor functions that makes you wonder if the standard, ultra-quiet testing environment in school is really ideal. Some (probably most) lightning calculators such as Inaudi, Colburn, Safford, and Benjamin, were or are quite agitated while performing. These are termed auditory calculators, but there are visual calculators as well (Diamondi, who had a “photographic” memory, Ruckle, Marathe, Dase, etc.) and those who don’t fit neatly into either category (Klein, Aitken). There are also those who experience synaesthesia, seeing colors when hearing or visualizing numbers (such as Daniel Tammet, who also visualizes “landscapes” and “spirals” of numbers. Smith even describes a “tactile” calculator.

Another topic of great interest and historical misunderstanding concerns the calculation process itself. It is often assumed that the results are spontaneously produced by an unconscious, mysterious and instantaneous process. Smith concludes that this is false, that the calculation proceeds through a sequence of operations that is conscious or semi-automatic, much like spoken language or touch typing. The brain scan figures that are shown later, in fact, show in red the areas of the brain used by the modern lightning calculator Rüdiger Gamm (in green and red) compared to several non-expert calculators (in green) as described in a paper at http://stepanov.lk.net/mnemo/gamm.html .

So how fast were/are these lightning calculators? The short answer is that the reported times, those that have any validity at all, are all over the map. Often those who report on times did not recognize attributes of specific problems that led to easy solution, or based their reports on second-hand or promotional material, or ignored delay tactics such as writing down or repeating the problem. Often the reports don’t indicate when timing began and stopped, and it often goes unreported whether the problem was in sight during the calculation and whether the answer was produced digit-by-digit or as a complete solution. Some times are considered beyond credibility or markedly inconsistent with the difficulties of various questions.

Smith attempts to sort through the available data on historical calculators, a seemingly frustrating enterprise. A decent set of tests were conducted by the noted psychologist Alfred Binet in 1894 and there was so much confusion on the best way to measure response times that a device that traced respiration on a revolving cylinder was settled on, but even Binet didn’t record whether the answers were written down or if the first or last digit of the answer was the initial trigger. Inaudi took an average of 2.0 seconds for 2-digit by 2-digit (2x2) multiplications, 6.4 seconds for 3x3 multiplications, 21.0 seconds for a 4x4 multiplication, and 40.0 seconds for a 6x6 multiplication. Diamandi did much worse but the problem was removed from view during the test and timing was definitely stopped after the last digit was written, so it is not a fair comparison. Ruckle and Finkelstein did worse than Inaudi in later identical tests. Klein was demonstrably fast, but in tests in 1953 he was allowed to view the problem and write down digits as he obtained them, a definite advantage for someone like him who used cross-multiplication. Nonetheless his time of 48 seconds to multiply two 9-digit numbers and 65-2/3 seconds to multiply two 10-digit numbers is very impressive. Klein also extracted integer roots of numbers, particularly 13th roots of 100-digit numbers, achieving a 1 min 28.8 sec time at one point.

As for the size of problems, Dase reportedly multiplied two 8x8 digit multiplications in 54 sec, a 20x20 digit multiplication in 6 min, a 40x40 digit multiplication in 40 min, and a 100x100 digit multiplication in 8 ¾ hours. Gauss later posed the question of who checked that last answer (adding that it was "a crazy waste of time"), and in fact lightning calculators often made errors, but of course those aren't typically reported. Buxton, who could not read or write numbers, squared 725,958,238,096,074,907,868,531,656,993,638,851,106 in his head over the course of 2-1/2 months, an astonishing feat of concentration that is scarcely marred by an error of one digit in the answer (this was an attempt to square 2^{139} but based on a flawed value for 2^{138}). Alexander Craig Aitken extracted non-integer roots among other calculations, merging his natural speed in arithmetic with mathematical approximation and iteration formulas. And John Wallis before him extracted the square root of $3x10^{40}$ to 21 digits during one sleepless night and of a 55-digit number to 27 places during another night. Dase is also reported to have extracted square roots of perfect squares of 100 digits and 60 digits, but with no times given.

The biographical details of these lightning calculators make interesting reading but are not the focus of this essay. There are a number of sites that provide the history of their lives or links to them, such as those listed here:

http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Mental_arithmetic.html
http://www.mentalcalculation.com/calculators/list.htm
http://www.nzedge.com/heroes/aitken.html
http://www.answers.com/topic/truman-henry-safford
http://www.math.buffalo.edu/mad/special/fuller_thomas_1710-1790.html
http://www.mentalcalculation.com/misc/bbc1954.html

and especially Oleg Stepanov's site that contains many historical articles on various mental calculators:

http://stepanov.lk.net/mnemo/mnemoare.html

There are modern-day lightning calculators, of course. Some of those listed above who have a good amount of history in this field (such as Arthur Benjamin and Shakuntala Devi) still perform in public. There are also relative newcomers that I am aware of through the Yahoo Mental Calculation Group, such as Rüdiger Gamm, Gert Mittring, Alexis Lemaire, Robert Fountain, George Lane, John van Koningsveld, Alberto Coto, Willem Bouman, Andy Robertshaw, Matthias Kesselschläger, Yusnier Viera Romero and Jorge Arturo Mendoza Huertas, but this is a highly Eurocentric view (other than the last two) because of the makeup of the Yahoo group and the participants in the Mental Calculation World Cup held in Europe. Unfortunately, I'm not familiar with the many other newer lightning calculators from around the world—Chan Hee Yi of Korea has been pointed out, and India certainly has a number of such talented individuals. (I nearly decided not to list any modern calculators to avoid slighting other people who certainly deserve to be listed—rest assured that all omissions are due to my limited knowledge in this area.)

Below are some links to good videos on lightning calculators (in various languages):

Wim Klein and Hans Eberstark at CERN (where Klein worked) from 1973:
http://video.google.co.uk/videoplay?docid=-3917808389759629434

Wim Klein at CERN in 1959
http://www.youtube.com/watch?v=KqQrJvPP9eo

A two-part documentary on Rüdiger Gamm:
part 1: http://www.youtube.com/watch?v=NUsD2V6ijyQ&feature=related
part 2: http://www.youtube.com/watch?v=oqxhxIuEGRw&feature=related

Arthur Benjamin:
http://www.youtube.com/watch?v=8Jb7m2vYZaQ
http://www.youtube.com/watch?v=M4vqr3_ROIk

A lecture by Gert Mittring on extracting cube roots:
http://www.youtube.com/watch?v=NC3DGyJG5ME

Jan van Koningsveld:
http://www.youtube.com/watch?v=rgrTQa--EX8

Willem Bouman on Dutch TV:
http://www.youtube.com/watch?v=bMnZNIn5ha0

The history of lightning calculators is interesting from a human standpoint, but it's perhaps more intriguing because the methods they learned or developed are uniquely suited for fast mental calculation. These methods are different from the ones taught in school for pencil-and-paper solution, and therefore most people are quite surprised when they find out that other algorithms such as these exist. Techniques designed specifically for mental calculation are the subject of the second part of this essay.

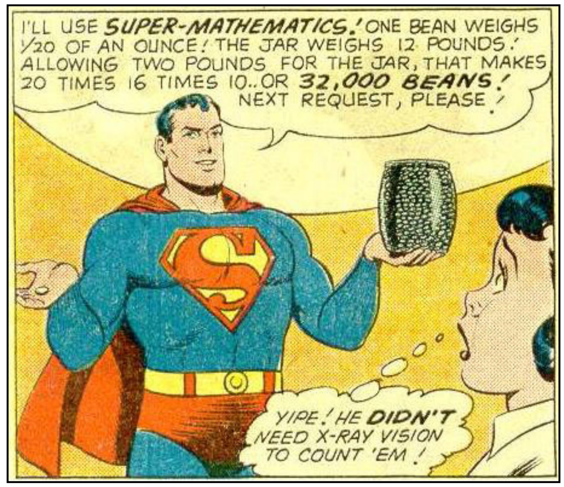
Part II: The Methods

The types of calculations performed by lightning calculators were historically quite limited, notable mainly for the size of the numbers and the speed at which they were manipulated. But remember that the questioner had to verify every calculation by hand, making higher powers and roots (particularly inexact roots) much less feasible. The dawn of calculators and computers propelled some of these tasks into hitherto uncharted territories such as 13^{th} or 23^{rd} roots, deep roots of inexact powers, and so forth, much of it supported by more sophisticated mathematics. Here we will review the methods of calculation used in the past, many of them not commonly known, as well as other techniques that are relatively new.

The traditional demonstrations of lightning calculators fall into the following categories:

- Fast addition of numbers (not that common, actually)

- Fast multiplication of multi-digit numbers (very common, and with more success than Superman demonstrates)
- Fast division (uncommon)
- Factoring of large numbers or finding them to be prime (common)
- Extraction of roots of perfect powers (very common)
- Extraction of roots of numbers that are not perfect powers (rare)
- Raising numbers to various powers (common)
- Finding logarithms of numbers (uncommon)
- Finding one or more sums of four squares that add to a given number (occasional)
- Calendar calculations (exceedingly common)
- Compound interest (isolated)



In addition, there are more modern methods that can be used, particularly for approximating logarithms, exponentials and trigonometric functions, that have been constructed for those interested in these types of problems.

The main techniques will be highlighted in sections below devoted to each of these tasks. It is important to realize that lightning calculators were highly individual in how they approached these tasks, and most calculators have such a vast knowledge of number facts that answers were often obtained immediately from memory or following only slight adjustment. As one example, Klein learned through experience the multiplication table through 100x100 and used it to great advantage doing cross-multiplication in 2-digit by 2-digit chunks. He also knew squares of integers up to 1000, cubes up to 100, and roughly all primes below 10,000. He also knew logarithms base 10 to 5 digits for integers up to 150.

Sometimes calculators used a mnemonic scheme, often of their own design, to aid in remembering these number facts. Mnemonics is the association of digits with images or letters in a sentence. Arthur Benjamin presents in his book, *Secrets of Mental Math*, the mnemonic scheme he uses to remember intermediate values during long mental calculations, based on a phonic method a few hundred years old. I ran across a chapter from a 1910 book that uses this same scheme to encode the cubes of all 2-digit numbers, and on a lark I modernized its quaint phrases and extended its scope to provide squares as well, and I wrote it all up in a paper found at

http://www.myreckonings.com/Dead_Reckoning/Online/Materials/Mnemonics_for_Squares_and_Cubes.pdf

But the more involved calculations also involve algebraic methods deduced by the performer through familiarity with the processes or, increasingly today, by consciously applying mathematical relations, number theory and numerical approximations. Some of the methods described below receive greater attention in Smith's book, while others are described in greater detail in other references. An excellent source for the world records in various categories of memorization and mental calculation can be found at

http://www.recordholders.org/en/list/memory.html

I might as well mention here that there is a movement to assign discovery of quite a few of these algebraic techniques to an ancient system of Vedic Mathematics rediscovered between 1911 and 1918 from the Sanskrit texts known as the Vedas by Sri Bharati Krsna Tirthaji (1884-1960) and expressed as sixteen Sutras. See http://www.vedicmaths.org/Introduction/History/History%20of%20VM.asp for an overview of these beliefs. For a detailed presentation of these Sutras as well as outright criticism of the supposed origin of them and their overall effectiveness as an educational tool, see http://arxiv.org/ftp/math/papers/0611/0611347.pdf . In my opinion, and I know this is not a popular one in some circles, systems such as this (and including resurgent schools for teaching the abacus and soroban in China, Japan and elsewhere) divert students' time in much the same way as the "New Math" introduced in U.S. schools in the 1950's and 1960's.

Fast Addition

It might seem that rapid addition would be a common demonstration for lightning calculators, but Smith notes that lightning calculators, driven by their interest in numbers, typically found addition and subtraction too dry for study. Inaudi and Bidder would add several multi-digit numbers, and there were a few more who specialized in just this task, but their methods were necessarily straightforward. The common theme seems to be grouping numbers into groups of digits to add separately, minding any carries or borrows as needed. I remember reading somewhere of a tip for adding a column of 3-digit numbers such as a grocery bill that proved a surprisingly helpful technique. In such a case it's easier to add the tens and ones digits as groups and then add the hundreds digits at the end. So say you are presented with a column of numbers such as

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245
814
152
 81
696
317
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Adding them as single digits can be slow and adding them as 3-digit numbers can be confusing, so we might add 45+14=59, add 52=111, add 81=192, add 96=288 (where 96=100-4), add 17=305. Then add 2+8+1+6+3=20 and with the carry of 3 we have 2305 as the sum. Now this might be a lot slower for you than just adding the columns in individual digits, but a practiced calculator can add 2-digit numbers in a flash (or 3-digit numbers and so on), so with some development this can be a faster alternative.

Fast Multiplication

I haven't heard of any lightning calculator who didn't or doesn't perform multiplications of multi-digit numbers. Many of them had such an intimate knowledge of factors and multiples built up after years of practice that often such a problem could be re-arranged into a known one plus some correction such as an additional factor. Some common products produce numbers that are easy to multiply by another number, so knowing such convenient products can be a real help. For example,

67 x 3 = 201	23 x 13 = 299	19 x 21 = 399
17 x 47 = 799	89 x 9 = 801	53 x 17 = 901
37 x 27 = 999	7 x 11 x 13 = 1001	23 x 29 x 3 = 2001
31 x 43 x 3 = 3999	and so forth…	

The two most common techniques used by lightning calculators for mental multiplication are ***adding partial products*** and performing ***cross-multiplication*** on the digits.

Partial products are the combinations of the individual digit multiplications, and they are added up from left to right to find the product:

46 x 58 = 40x50 +40x8 + 6x50 + 6x8
= 2000 + 320 + 300 + 48
= 2668

The terms are added as they are calculated, so when 40x8 is calculated, it is added to 2000 to get 2320, then 6x50 is added to get 2620, and finally 6x8 is added to yield 2668. There is only one running total to remember.

Cross-multiplication does not involve these large sums. The digits of the product are found one at a time, but the procedure has the disadvantage that the digits are produced from right to left, so they must be remembered and reversed to recite the answer verbally. Typically the digits are written as they are obtained from right to left. In this method the combinations of single-digit products that contribute to each digit of the result are added, including carries. For example,

46 x 58:
6x8 = 48, or **8** with a carry of 4
4x8 + 6x5 + 4 = 66 , or **6** with a carry of 6
4x5 + 6 = **26**
Answer: 2668

Both of these methods have the advantages that they can produce results very quickly with practice, they scale up very well with larger multipliers, and they don't require any multiplications beyond one-digit by one-digit (Klein used 2-digit by 2-digit cross-multiplication). They are simple and very practical methods.

There are various ways to simplify multiplication based on the properties and relationships of the numbers involved. We might notice in a problem that one of the multipliers is quite near a very round number, say, a multiple of 10 or 25. We can multiply by that round number instead and adjust for the difference at the end. For example,

29 x 34 = 30x34 − 34

To find 30x34 here, we would multiply from left to right: 30x30 + 30x4. Now if a multiplier exceeds a multiple of 10 by the amount of the multiple, we can use the multiple of 10 and add 1/10 of that result. If a multiplier lies below the multiple of 10, we subtract 1/10 of the result. Multiples of 11 and 9 have these properties.

33 x 62: Find 30x62 = 1860, then 1860 + 186 = 2046
36 x 62: Find 40x62 = 2480, then 2480 − 248 = 2232

We would not subtract 248 directly in the last example, but rather subtract 250 and add 2, a slightly different view of subtraction that makes a large practical difference.

We can also look at a number as a collection of convenient groupings. For example, we can multiply 124726132 by 5 by first halving each even grouping in the first number and then appending zero:

12 4 72 6 132 x 5 = 6 2 36 3 066 0 or 8 32 6 31 x 5 = 4 16 3 15 5

Multiplying a number by 15 can be done by multiplying by 10 and adding half the result. We can think of adding a zero, and then adding half of each even grouping to itself, working left to right and keeping the same number of digits in the grouping as it started with. If a grouping ends up with an additional digit, the upper digit is added to the grouping to the left. The presentation below makes the calculation look more difficult than it actually is—the result is generated smoothly from left to right, with perhaps a correction for a carry from the next grouping, as with the carry of 1 from the (72+36) grouping below to the group on its left:

12 4 72 6 132 x 15 =	(12+6)	(4+2)	(72+36)	(6+3)	(132 + 66)	0
=	18	7	08	9	198	0

Multiplication by 25, or 100/4, can be thought of as appending two zeros and dividing by 4. Multiplying by 50 can be done as 100/2, 75 as 300/4, 125 as 1000/8, and so forth.

These are reasonable and readily understood concepts that involve looking at the whole number rather than individual digits. This is a mental shift that is subtle but critical in developing a number sense. Methods like these are also more general than they seem at first, because if they *almost* apply, we can use them on nearby numbers and then apply a correction at the end.

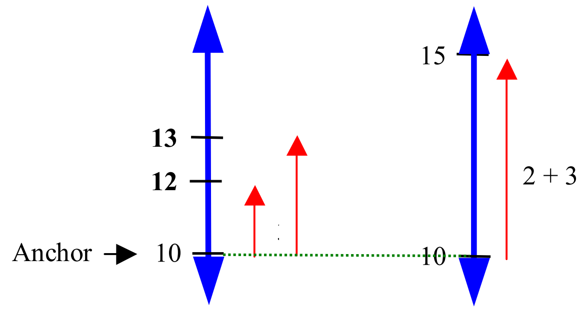
These methods all involve thinking about the properties of numbers, so they appeal to me as methods for somewhat specific circumstances. However, there is a type of method that is useful in a very wide variety of multiplications. When the multipliers are a distance **c** and **d** from a round number, their product can be represented by the product of the round number and the sum of the round number and the two differences, with the product of the two differences added at the end as a small correction. There does not seem to be a consistent name for this method in the literature; I call it the *Anchor Method*:

$$(a+c)(a+d) = a(a+c+d) + cd$$ ***Anchor Method***

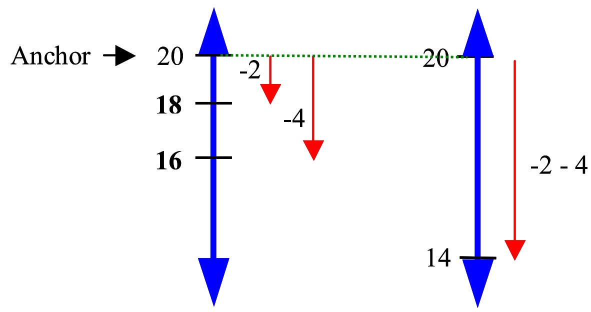
This is much easier to use than it might appear, as we will see, and a knack for it is easily developed with a small amount of practice. The concept can be taught to children. I visualize "anchoring" one multiplier at the round number, and then literally stringing out the differences from the original numbers from this anchor to find the other multiplier. It will turn out that the original multipliers move outward, their product will be less than the original, so the correction at the end needs to be added, and if they move inward, the correction is subtracted. This corresponds to the intuitive (and correct) concept that a square has the greatest area for a given sum of side lengths; the rectangle produced by shifting length on a square from one side to another side will have a smaller product of the two sides because $(x+n)(x-n) = x^2 - n^2$ is always less than x^2.

Below are three representative problems and a visualization of each solution. (The numbers are shown on vertical number lines because I "see" number lines as vertical rather than horizontal. I remember having difficulty learning the number line concept in grade school, and I believe it was due to think a vertical layout would be much more intuitive to children (and me) who think numbers go up as they get higher.)

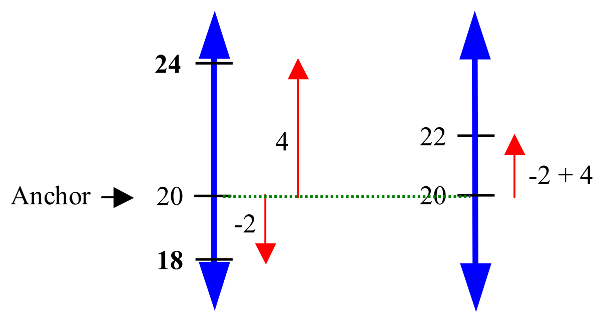
$12 \times 13 = 10 \times 15 + 2 \times 3 = 156$



$18 \times 16 = 20 \times 14 + 2 \times 4 = 288$



$18 \times 24 = 20 \times 22 - 2 \times 4 = 432$



An anchor of 100 is very common, say, $84^2 = 100 \text{x} 68 + 16^2$. With 100 as the anchor, we can find 68 as the last digits of 84 doubled rather than by finding the difference between 100 and 84 and subtracting this from 84.

If the numbers to multiply are far apart, though, we can end up with a large correction term **cd**. There are a few strategies to bring the multipliers nearer to each other:

1. Subtract one number from a very round number (or add it to a very round number) to bring it closer to the other number:

   $$23 \text{ x } 67 = 23(100-33) = 2300 - 23\text{x}33 = 2300 - (20\text{x}36 + 3\text{x}13)$$

2. Divide or multiply one number by a low integer and add a correction:

   $$23 \text{ x } 67 = 23\text{x}33\text{x}2 + 23 = 2(20\text{x}36 + 3\text{x}13) + 23$$

3. Break one number into two convenient parts:

$$23 \text{ x } 67 = 23(50+17) = 2300/2 + 23\text{x}17 = 1150 + 20^2 - 3^2$$

In the end we can use our creativity and experience to manipulate the calculation as we wish.

One of the most powerful tools in mental calculation is converting the multiplication of two different numbers into the square of the average minus the square of the distance to the average. This is shown by the *Midpoint Method*, an algebraic identity:

$$(a+c)(a-c) = a^2 - c^2$$ ***Midpoint Method***

where **a** is the average of the two numbers, **(a+c)** is one of the numbers, and **(a−c)** is the other number. This is algebraically equivalent to the Anchor Method formula if **d** = **-c**, or in other words when the anchor is midway between the two multipliers. The choice of the anchor as the midpoint or some other number depends on the problem and on personal preferences, but there is no doubt that using the midpoint is a very common technique. For example,

$28\text{x}32 = 30^2 - 2^2$
$52\text{x}78 = 65^2 - 13^2$

or, considering the first problem in this section,

$46 \text{ x } 58 = 52^2 - 6^2$

Less convenient multipliers can be manipulated in a number of ways to use this technique. We might have the case where there is no midpoint of the two multipliers—here we can adjust one of the multipliers by 1, do the calculation, and then provide a correction to account for the original adjustment, as for $28\text{x}33 = 28\text{x}32 + 28 = 30^2 - 2^2 + 28$, but in this particular case it may be easier to use the Anchor Method from the last section: $28\text{x}33 = 30\text{x}31 - 2\text{x}3$.

To calculate squares we might use the Midpoint Method *in reverse*. We can split a square into the product of two numbers equidistant from the original number, and add the square of that distance, again one scenario of the Anchor Method. For example, let's continue with one of our examples from earlier:

$52\text{x}78 = 65^2 - 13^2$

Now we find 65^2 by spreading 65 in both directions by an equal amount and adding the square of that amount. Here a good spread is by 5, yielding $65^2 = 60\text{x}70 + 25 = 4225$. Similarly, $13^2 = 10\text{x}16 + 9 = 169$. So we can turn a general multiplication into a square plus a small correction, and we can turn that square into an even simpler multiplication and one more small correction if needed. Again, I find it helpful to remember that the average squared will always be larger than the spread numbers multiplied, so when collapsing two multipliers to a square you *subtract* the correction, and when spreading a square to the product of two numbers you *add* the correction. These transformations become automatic and very fast after a bit of practice.

Many of you may recognize in the example of 65^2 the trick for squaring numbers ending in 5: multiply the number left of the units digit by that number plus one, and then append 25, as in 6x7 | 25 = 4225. Now we can see why that works.

The Midpoint Method described earlier applies to larger numbers, e.g., $244 \times 376 = 310^2 - 66^2$. But 310^2 is really just a square of a two-digit number followed by two zeros—what if we had ended up with a three-digit square here? Again we split the square into two numbers equidistant from the original number, adding the square of that distance. To illustrate, $244 \times 382 = 313^2 - 69^2 = [300 \times 326 + 13^2] - 69^2$, and we end up with a simple calculation if we know the two-digit squares.

And there are indeed a variety of other techniques for finding squares. Most of these involve expressing the number to be squared as the sum of two other numbers that are more easily squared, using the *Binomial Expansion for Squares*:

$$(a+b)^2 = a^2 + 2ab + b^2$$ ***Binomial Expansion for Squares***

To illustrate,

$$34^2 = (30+4)^2 = 30^2 + 2 \times 30 \times 4 + 4^2 = 1156$$
$$69^2 = (70-1)^2 = 70^2 - 2 \times 70 \times 1 + 1^2 = 4761$$
$$313^2 = (300+13)^2 = 300^2 + 2 \times 300 \times 13 + 13^2 = 90000 + 7800 + 169 = 97969$$

In another application of the binomial expansion, one of the most intriguing and useful techniques easily finds the square of a number near 50. Here we add the difference from 50 to 25, multiply by 100, and add the difference squared. If the number is within 10 of 50, we can add the difference to 25 and simply append the distance squared rather than adding it. Let's use the vertical bar "|" to separate two-digit groups. Note that if we end up with a 3-digit result in a grouping, its most significant digit would be added to the group to its left. In this notation,

$$(50+a)^2 = (25 + a) \mid a^2$$

Ex: $52^2 = (25+2) \mid 2^2 = 2704$
$44^2 = (25-6) \mid 6^2 = 1936$
$38^2 = (25-12) \mid 12^2 = 13 \mid 144 = 1444$

This is a simpler way of thinking of the binomial expansion $(50+a)^2 = 2500 + 100a + a^2$.

We can also use the fact that multiples of 25 are fairly round numbers. We can square numbers near 25 using the expansion $(25+a)^2 = 625 + 50a + a^2$, as $27^2 = 625 + 100 + 4 = 729$. The relation $(75+a)^2 = 5625 + 150a + a^2$ can be used to find, say, $78^2 = 5625 + 450 + 9 = 6084$. We can reformat these into our notation, noting that a .5 in a group is converted to a 50 in the group to the right of it:

$$(25+a)^2 = (6 + a/2) \mid (25 + a^2)$$ *Ex:* $27^2 = (6+1) \mid (25 + 2^2) = 729$

$$(75+a)^2 = (56 + a + a/2) \mid (25 + a^2)$$ *Ex:* $78^2 = (56+3+1.5) \mid (25 + 3^2) = 60.5 \mid 34 = 6084$

Alternatively, we can re-arrange the binomial expansion of two-digit squares ending in 9, 8, or 7 in another interesting way:

$$(10a+9)^2 = 100a(a+1) + \mathbf{8}0(a+1) + \mathbf{1}$$
$$(10a+8)^2 = 100a(a+1) + \mathbf{6}0(a+1) + \mathbf{4}$$
$$(10a+7)^2 = 100a(a+1) + \mathbf{4}0(a+1) + \mathbf{9}$$

where the digits in bold comprise the square of the units digit. So 79^2 = 5600 + 640 + 1 = 6241, 87^2 = 7200 + 360 + 9 = 7569, and so on.

If a neighbor of the number has a square that is known or easily calculated, we can use this convenient square and adjust for the difference. Since $(a+1)^2 = a^2 + a + (a+1)$, we can find $31^2 = 30^2 + 30 + 31 = 961$. Similarly, $29^2 = 30^2 - 30 - 29 = 841$. For other neighboring numbers we can find the square of the convenient number, then add or subtract the original number, the final number, and twice each number in between, so $32^2 = 30^2 + 30 + 2x31 + 32 = 1024$, a square that we recognize from powers of 2. Ultimately we will find that the field is quite crowded for squaring numbers less than 100, and in a surprising development we eventually start looking to three-digit numbers for more interesting challenges.

Three-digit numbers can be treated like two-digit numbers in all these methods if we treat the leftmost two digits as a single digit, as in using the technique for squaring numbers ending in 5 to find 235^2 = 23x24 | 25 = 55225. We can also alter some of the methods slightly for three-digit calculations. The square of a number near 500 can be found by adding the difference from 500 to 250 and appending the difference squared as a three-digit group delineated by a comma:

$$(500+a)^2 = (250 + a) \;,\; a^2$$

so,

$$513^2 = 263{,}169$$
$$492^2 = 242{,}064$$

Multiplying larger numbers extends these rules further with a corresponding increase in difficultly. A more recent method of multiplying two 4-digit numbers is discussed in the Newer Methods section of this essay. I might add that an excellent, free training program for practicing multiplications up to 4x4 can be found at http://www.buildquiz.com/speed_math.swf .

Fast Division

Division was not a common task except in the limited context of factoring a number, which is not really division in the truest sense. When this was done, as in the case of decimalizing a fraction, it was often done by reversing known multiplications or by taking advantage of properties of division by small integers (which might be factors of the actual divisor).

There are some properties of reciprocals 1/t that help in finding their decimal expansions, and of course a calculation of s/t might first calculate 1/t and then multiply the answer by s. For a denominator t with prime factors of 2 and 5 only, the number of decimal places in its decimal expansion will equal the highest power of 2 or 5, so any s/16 will terminate after the fourth decimal place since $16 = 2^4$ x 5. If t is a prime number, the decimal expansion of 1/t will consist of some zeros followed by repeated groups of digits. The length of this group will be (t-1) or a factor of (t-1), the first type occurring for t = 7, 17, 19, 23, 29, 47, 59, 61, 97, … So 1/7 will have a 6-digit repeating group and in fact 1/7 = 0.142857142857142857… When t is one of these special primes, the corresponding digits in the two halves of the group will add to 9, so here if we find 1/7 to three places (0.142) we immediately know the next three digits (857) and we now have the whole repeating group. A numerator here other than 7 that is less than 7 simply rotates the digits of the repeating group, maintaining this relationship. A numerator greater than 7 will consist of some digits to the left of the decimal point, followed by the repeating groups based on the remainder. Also, the repeating group of 1/t for a prime t with a units digit of 1 will have a last digit of 9 and vice-versa, otherwise the last digit of the repeating group will be the same as the last digit of t. So for 1/7 we know immediately that the last digit of the repeating group will be 7, so we take the reciprocal to 2 places (0.14), then we know the next digit is (9-7) or 2, then we complete the entire group as 142857.

There are many such relationships that make such divisions faster and simpler. People will generally request division by prime numbers anyway, so it's possible to memorize some of these repeating groups (or half of each group). Some questioners are aware that 1/97 has 96 digits in the repeating group and ask for that reciprocal. Aitken remarked that this was sometimes asked and then rattled off the answer, and we will see a critique in the *Media* part of this essay of such a question posed in a modern documentary.

Aitken used these properties of reciprocals to decimalize fractions, but he also would use straight division but with a simpler divisor, making corrections as needed in each step. For example, when a divisor ended in 9, such as 1/59, he would divide instead by 60 as described in his paper at http://stepanov.lk.net/mnemo/aitkene.html :

6)1.016949152 = .0169491525…

where the adjustment for the simpler divisor amounts to adding the previously obtained digit to the next digit in the dividend (here always 0).

If there is situation that involves dividing by, say, a four-digit value, we can try to reduce the denominator to an integer of one or two digits at most, as short division by numbers of this size are not too difficult. First, we convert the denominator to an integer by shifting its decimal point and shifting the decimal point in the numerator by the same amount. For example, 4.657/.07 = 465.7/7 = 66.53 to four digits. Then we look to simplify the fraction by dividing the numerator and denominator by low common factors. For example, .2420/7.2 = 2.420/72 = .605/18 = .0336 to four digits. We could have twice divided through by 2, but the last two digits of both numbers are divisible by 4, so the entire numbers are divisible by 4. The division by 18 can be done directly (I would count up by 18's here, so for 60 we have 18→36→54 gives 3 remainder 6, then for 65 we know 54 again gives 3 remainder 11, then for 110 we double 54 to give 6 remainder 2, etc.), or we can divide .605 by 2, then by 9. Division by 2 is easiest if the number is split into even number groups, so .605 is split into .(60)(50), so half of each even group gives .3025, and dividing this by 9 yields .0336 as before. In other words, we can divide the denominator by a convenient factor even when the numerator is not evenly divisible by it, e.g., 35/36 = 5.833/6 = .9722 .

We can also adjust the denominator a little bit to get it to a round number as long as we adjust the numerator by the same percentage. If we are solving 247/119, we see that the numerator is about twice the denominator, so if we adjust 119 up to 120, we need to adjust 247 by about 2, and we arrive at 249/120 = 24.9/12 = 2.0750 compared to the actual value of 2.0756… With experience, we might notice that 247 is twice 119 plus about 10%, so we could add 2.1 to 247 to get a more accurate 24.91/12 = 2.0758. If we have 91.5/353, we can adjust the denominator down to 350 and double the fraction to have a single-digit division, so 91.5/353 = 90.75/350 = 181.5/700 = 1.815/7 = .2593 , where we reasoned that decreasing 353 by 3 was roughly equal to decreasing 91.5 by ¾ . Our answer will be a bit high, since 91.5 is a bit more than 1/4 of 353, so we might subtract a tiny bit from our answer (which is in fact in excess by .0001). This shifting technique may not seem like much, but as a graduate teaching assistant I impressed more than one physics class by using it to mentally calculate answers to problems.

Finally, we can generalize an approximation that is valid for small b, that is

$1 / (1+b) \approx 1-b$

to get

$a / (c+b) \approx (a/c)(1 - b/c)$

$a / (c-b) \approx (a/c)(1 + b/c)$

The error here is about .01 of a/c when b/c is 1/10, and about .0001 of a/c when b/c is 1/100 , low for both approximations. This is a nice alternative to shifting the denominator when the numerator is not a simple multiple or fraction of the denominator. For example, 27/61 ≈ (27/60)(1 – 1/60) . Here we can find 27/60 = 2.7/6 = .4500 , then subtract .4500/60 = .0450/6 = .0075 to get .4425 compared to the actual value of .4426 . Since 1/60 is 1/6 of 1/10 and the error follows a square law, we are low by about (.45/36)(.01), or .0001 , but this is for better calculators than me.

In short, long division should not be as intimidating as it might seem, particularly since we have flexibility in our accuracy. If the problem is difficult to rearrange, we settle for less accuracy; if it can be easily manipulated, we take what we are offered.

Factoring and Primality Testing

Factoring, I imagine, would have fired the imagination of a lightning calculator. Here every carefully preserved number fact, every trick in the book could be thrown at the problem in a wild attempt to unlock the puzzle in a highly-charged atmosphere of anticipation. If an answer were to emerge immediately, either through luck or a creative leap, the solution is recorded as an instance of true genius. The fact that methods of factoring are not commonly known, and that no closed form method of factoring exists in general, lends this feat an aura of mystery that high-order roots once had prior to calculators. In fact, determining whether a number has **no** factors (other than 1 and itself, of course)—or in other words declaring a number to be prime—is more difficult than finding factors of a compound number. This one category may be the true measure of the depth and brilliance of a mental calculator. Many were adept at it, including Klein and Aitken. At 8 years of age, Zerah Colburn could factor 6-digit numbers or declare them prime.

Trial and error was the most common method, but only after reducing the possible factors of the number to a minimum. This can be done by looking at the last few digits of the given number and having memorized products that end in those numbers. As trivial examples, an even last digit such as 0,2,4,6, or 8 is obviously divisible by 2, and a last digit of 0 or 5 is divisible by 5. If the last two digits of a number are divisible by 4, the number is divisible by 4, and if the last three digits are divisible by 8, the number is divisible by 8. There are a number of divisibility tests for small primes that are commonly known (see http://en.wikipedia.org/wiki/Divisibility_rule). Beyond that, lightning calculators often knew all products of two numbers that would end in any two digits, and it's a good bet that they knew a lot that ended in various 3-digit numbers. These would significantly limit the number of factors to verify by multiplication, which only have to be tested up to the square root of the given number, and unless the number turns out to be prime there is no need to test every one before a true factor is found.

The mathematician Fermat produced the first methodical method of finding factors of integers. Since $a^2 - b^2 = (a+b)(a-b)$, then if two squares can be found whose difference equals the given number, two factors will have been found. As mentioned in the *Fast Multiplication* section, squaring numbers is generally easier than multiplying two numbers, and calculators could also memorize tables of last digits of differences of squares to limit these possibilities as well. To simply test whether a number of the form (4n+1) is prime, possible **sums** of two squares could be checked, as a prime of this form can only be expressed as a single such sum (Smith reports that Aitken and Klein used this fact).

Factoring is a fun diversion. I know at least two people who practice factoring car license plates or the last few digits of the odometer while driving. Not recommended.

Integer Roots

Producing high-order roots of perfect powers is extremely common, generally possessing all the drama of factoring or primality testing (and assuredly more) without the nuance or difficulty of the latter. It makes great press, though (see the later discussion on the media). I say "generally" because at the highest levels of this task, a distinction lost on the public, a calculator does have to stretch his/her capabilities in remarkable ways to find the answer. Klein was an expert on this, along with Dagbert and Marathe, but it's safe to say that integer roots were asked of all lightning calculators, then and now.

As a good rule of thumb, the difficulty of extracting a root does **not** depend on the order of the root (unless it is an even root, which is rarely asked) but rather on the **number of digits in the answer**. This is critical in any evaluation of such a feat—remember this the next time you hear that someone extracted the cube root of a number near a billion, or the 13th root of a 39-digit number or the 23rd root of a 69-digit number, all of which have at most 3-digit answers.

It turns out that the last digit of a root of an order (4k+1), such as a 5th root, a 9th root, etc., is the same as the last digit of the power, so for example the 13th root of 79,469,020,066,571,739,979,222,359,560,551,645,783 has a last digit of 3. The last digit of a root of an order (4k+3), such as a cube root, a 7th root, etc., is different but unique compared to the last digit of the power, so with some memorization the digit-pairing is also known for these roots. Between these two rules we have one of the three digits of an exact, odd root.

Now the highest digit can be found by memorizing the ranges of powers for the various starting digits 0 through 9. In our example above, of the 13th root of 79,469,020,066,571,739,979,222,359,560,551,645,783 we might have memorized the fact that the 13th power of a 3-digit number starting with 8 ranges from $55x10^{36}$ to $250x10^{36}$, and here we have $79x10^{36}$, so we now immediately know the number is of the form 8n3, where n is the final digit to determine.

We can use remainders from divisibility tests to find the missing middle digit. For example, the remainder after a cube is divided by 11 (the 11-remainder) is uniquely paired with the remainder after its cube root is divided by 11, as

(0,0), (1,1), (2,8), (3,5), (4,9), (5,4), (6,7), (7,2), (8,6), (9,3), (10,10)

We can find the 11-remainder by subtracting the sum of the even-place digits of a number from the sum of the odd-place digits, then adding or subtracting multiples of 11 to find a number between 0 and 10. For example, if we have a cube 300763, the 11-remainder is (3+7 +0) – (6+0+3) = 1. Therefore the 11-remainder of the cube root is, from the pairing, also 1. We know from earlier than a cube and its root has a unique pairing of last digits, which we can also memorize as

(0,0), (1,1), (2,8), (3,7), (4,4), (5,5), (6,6), (7,3), (8,2), (9,9)

So the last digit of the cube root must be 7 since 300763 ends in 3, and since the cube is less than a billion it is a 2-digit number n7. The 11-remainder (7-n) must equal 1, so n = 6 and we find the cube root of 300763 to be 67.

The 9-remainder can be tried for fifth roots, as it produces (power, root) pairs of

(0,0), (0,3), (0,6), (1,1), (2,5), (4,7), (5,2), (7,4), (8,8)

The only ambiguity is when the 9-remainder of the fifth power is 0, and in this case the 11-remainder can then be used to distinguish them.

In our running example of the 13th root of 79,469,020,066,571,739,979,222,359,560,551,645,783 the 13-remainder will be the same as the 13-remainder of the root. We could do short division by 13 working from left to right one digit at a time, or since 7x11x13=1001, we can divide out multiples of 1001 from the original number by subtracting each thousands group from the thousands group to its right:

79,469,020,066,571,739,979,222,359,560,551,645,783
390,020,066,571,739,979,222,359,560,551,645,783
(-370),066,571,739,979,222,359,560,551,645,783
436,571,739,979,222,359,560,551,645,783
135,739,979,222,359,560,551,645,783
604,979,222,359,560,551,645,783
375,222,359,560,551,645,783
(-153),359,560,551,645,783
512,560,551,645,783
48,551,645,783
503,645,783
142,783
641

Dividing 641 by 13 we arrive at a 13-remainder of 4. So 8n3 must leave 4 as a 13-remainder, and it doesn't take long to find the middle digit to be 2 and we have found the 13th root of a very large number indeed.

With some additional memorization of two-digit endings of powers it's possible to get the last **two** digits for a given root (and calculators often specialize in certain orders of roots), and this is also possible to do for the first two digits. This provides the ability to find roots of greater numbers of digits. Klein also used logarithms he memorized to calculate the first five digits of the answer, which also increased his range—he is an example of someone who has raised the bar on these calculations to extremely impressive heights. Alexis Lemaire, a present-day lightning calculator is another—his specialty is finding the 13th root of 200-digit numbers, which contains up to 16 digits.

I might add that I use the variable precision arithmetic (*vpa*) command in the MATLAB software package to generate arithmetic results to many digits. Here the command "vpa 823**13 200" will provide all digits of 823^{13} up to a maximum of 200 digits. Octave (found at http://www.gnu.org/software/octave/) is a free open-source alternative to MATLAB that is designed to accept MATLAB commands.

Non-Integer Roots

Irrational roots, that is, decimal roots of number that are not perfect powers, is historically rare, although it is more popular today because it is easy to use calculators and computers to generate problems. Aitken was able to approximate square and cube roots using numerical approximation techniques he was aware of as a mathematician.

Aitken could find the square roots of non-squares to five significant digits in about 5 sec. From an initial approximation n (a decimal or fraction) of the square root of a number N, he used the Newton-Raphson method for iterating a function to find a correction as $(N - n^2) / (2n)$. So for N=85, we can estimate the square root as 9 and find a better answer as $9 + (85 - 9^2)/18 = 9.22$ compared to the actual value of 9.219544… A closer initial value yields a much closer answer, so if we can do two-digit multiplications and divisions we can take an initial estimate of 9.2 to find a better answer of $9.2 + (85 - 9.2^2)/18.4 = 9.219565…$

Cube roots can be approximated by a similar mechanism—for a description (and for more examples of square roots) I heartily recommend reading Smith's book or Aitken's 1954 talk on mental calculation found at http://stepanov.lk.net/mnemo/aitkene.html .

If logarithms and anti-logarithms can be mentally calculated, this provides a different way of approximating roots, even higher-order roots. The 12th root of N, for example, can be calculated by finding log N, dividing by 12, and then finding the antilogarithm, all at whatever accuracy the calculator can produce. Klein used his memorized logarithms and simple interpolation to do this.

Powers of Integers (Involution)

As we saw earlier, squares and cubes of numbers offer advantages to the calculator. In general it's easier to use the binomial expansion of $(a+b)^n$ for a round number "a" and a small correction "b" than to multiply the number (a+b) by itself n times.

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
etc.

When multiplying, Mondeux would factor problems if possible, and if this reduced the problem to powers such as squares and cubes, he would employ the binomial expansion. In 1952 Klein raised 87 to the 16th power, which Smith assumes was most likely by successive squaring. Marathe is an expert on raising single digits to powers up to 20 (but how many different results is this, really?). Euler is reported to have mentally calculated the first six powers of all numbers less than 20 in one restless night.

Logarithms

Logarithms occur naturally in formulas, and as we have seen earlier they can be used to find roots of any order, even fractional roots, if there is a means of finding the antilogarithm as well.

We saw earlier than Klein had memorized the logarithms of the first 150 integers to 5 digits. Bidder had memorized those for the first 100 integers to 8 digits. By factoring a number and scaling the factors by multiples of 10 when needed, corresponding logarithms can be added to find the logarithm of the answer. For example, log 483 = log (3 x 7 x 23) = log 3 + log 7 + log 23 = 0.47712 + 0.84510 + 1.36173 = 2.68395. But what about log 487? Well, 487 = 483(1+ 4/483) ≈ 483(1 + 1/120) so log 487 ≈ log 483 + log(1 + 1/120). The well-known power series for the natural logarithm (denoted by ln) of a value 1+x for $x \leq 1$ and $x \neq -1$ is:

$$\ln (1+n) = n - n^2/2 + n^3/3 - \ldots$$

We can truncate this series very quickly if n is small. To find the common logarithm (to base 10) rather than the natural logarithm (to base e), we have to multiply ln(1+n) by log e = .4343. So

$$\log(1 + 1/120) = .4343 \ln (1 + 1/120) \approx .43/120 = 0.00358$$

Adding this value to our earlier result log 483 = 2.68395 we arrive at log 487 ≈ 2.68753 which is actually correct to the last digit shown.

There are other methods as well. Bidder used the following relation to arrive at the last correction above:

$$\log (1 + n) \approx 10^m \, n \log (1+10^{-m})$$

where m is chosen so that 10^m n lies between 1 and 10. Bidder memorized the values of log $(1+10^{-m})$:

log 1.01 = 0.00432…
log 1.001 = .000434…
log 1.0001 = .0000434…
etc.

where the digits approach log e = .4343… as m increases. The correction above was log (1 + 1/120) = log (1 + .00833), so m=3 will give 10^3 x .00833 = 8.33 and multiplying this by log 1.001 = .000434 we arrive at .0036 if we simplify the multiplication to 2 places. This is quite near the correction we calculated by our last method.

Note that n can be positive or negative in these relations, so the relations are useful when it is easier to find the logarithm either above or below the desired number.

Sum of Four Squares

Every positive integer can be written as at least one sum of four squares, so this task was occasionally asked of lightning calculators, particularly those who specialized in it such as Ruckle, Finkelstein and Klein. As a typical case, Ruckle expressed 15663 as a sum of four squares in 8 sec, followed immediately by a second sum. The same was done for 18111 in 26.5 sec and 63.5 sec, and for 53116 in 51 sec immediately followed by a second sum.

Like factoring, a solution for reducing an integer to the sum of four squares cannot be expressed in closed form, and success relies in part on the experience and cleverness of the calculator. I have written a paper summarizing methods for such reductions (a subject not covered by my book, by the way) that can be found at:

http://www.myreckonings.com/Dead_Reckoning/Online/Materials/Sum%20of%20Four%20Squares.pdf

In a different vein related to squares, given an integer c Mondeux could find two squares a and b that have a difference of c. He apparently knew that if $d = a - b$, then $b = (c - d^2) / (2d)$. Then it becomes a matter of finding d such that b is a positive integer, whence $a = b + d$. If c is odd then he could set $d = 1$ and then $b = (c-a)/2$ and $a = b + 1$.

Calendar Calculations

Calendar calculations are probably the most commonly performed feat of calculators, particularly aspiring calculators, but this happens to be my least favorite task. It usually involves finding the day of the week for any day in history, which has to take into account leap years and the Gregorian calendar change (which was adopted in various years by various countries, actually). Since this is an area I haven't studied in detail, I will simply provide some good websites that describe calendar algorithms:

http://rudy.ca/doomsday.html
http://calendars.wikia.com/wiki/Calculating_the_day_of_the_week
http://www.terra.es/personal2/grimmer/
http://www.cs.usyd.edu.au/~kev/pp/TUTORIALS/1b/carroll.html
http://litemind.com/how-to-become-a-human-calendar/

An algorithm for mentally computing the phase of the moon with 2-day accuracy between 2000 and 2009 can be found at
http://www.moonstick.com/in_head_2000-2009.htm

Compound Interest

Bidder mentally calculated simple interest on money at 10 years old and compound interest later in life as described in his 1856 talk found at http://stepanov.lk.net/mnemo/biddere.html . This interest is periodically compounded rather than continously compounded, which would require calculating exponentials. I am not aware of any other historical calculator who dealt with this area of mathematics.

Newer Methods

New methods for calendar calculation seem to appear now and then, and I presume these are being used by some. My book from 1993 also contains quite a few algorithms invented or adapted for mental calculations to high precision. In addition I have written quite a few papers on methods—the papers linked below reside on the Online Materials page of the section of my main

website devoted to the book
http://www.myreckonings.com/Dead_Reckoning/Online/Online_Material.htm .

For example, in the *Logarithms* section above we saw a method of calculating the logarithm of a number based on a nearby round number N whose logarithm is much easier to find. However, there are various other approximation schemes for finding such a logarithm. The most generally useful one of these, I think, is the following relation for n small compared to N:

$$\ln (N + n) \approx \ln N + 2n/(2N + n)$$

Compared to the problem in the earlier *Logarithms* section, the correction term to add to log 483 to find log 487 when using this new formula is .4343(8/970) = 0.00358, accurate as before to the last digit. If additional digits were taken for ln N and for this correction term, it would be found to be more accurate than the earlier result obtained by truncating the first term of the power series.

It's also possible to extend the Newton-Raphson method to churn out digits of a square root one or two at a time indefinitely, or at least until the calculator has reached their limit of time or ability. It's not necessary to read the book for this, as the method is provided in full in the following papers:

http://www.myreckonings.com/Dead_Reckoning/Chapter_3/Materials/Another_Square_Root_Example.pdf
http://www.myreckonings.com/Dead_Reckoning/Chapter_3/Materials/Square_Root_of_121432.pdf
http://www.myreckonings.com/Dead_Reckoning/Chapter_3/Materials/Alternate_Derivation_of_Square_Root_Algorithm.pdf

Manny Sardina has produced approximation algorithms for integer and fractional roots of numbers based on continued fraction representations:

http://www.myreckonings.com/Dead_Reckoning/Online/Materials/General%20Method%20for%20Extracting%20Roots.pdf

There are also various methods for calculating exponentials that were not used historically by mental calculators. The most promising one from the book is detailed in the following paper:

http://www.myreckonings.com/Dead_Reckoning/Chapter_4/Materials/Bemer_Exponentials.pdf

John McIntosh has discovered another method for exponentials that only requires knowing log 2 = .3010300 and log 3 = 0.477121. His presentation of this can be found at

http://www.urticator.net/essay/6/641.html

Algorithms for approximating trigonometric functions are also presented in the book. And in an odd grouping of functions sharing a similar approximation technique, I have written a paper that describes methods for mentally calculating the tangent, hyperbolic tangent, exponential and logarithmic functions to high accuracy:

http://www.myreckonings.com/Dead_Reckoning/Online/Materials/Fast_Approximation_of_Elementary_Functions.pdf

Finally, although lightning calculators historically could find products of two 4-digit problems or more (Zerah Colburn at 7 years old could multiply two 4-digit numbers), I wrote a paper on what I believe is an easier way to perform this task, one that is particularly useful when the difference between the first half and the second half of one of the numbers is small:

http://www.myreckonings.com/Dead_Reckoning/Chapter_2/Materials/A%20Method%20for%204x4%20Digit%20Mental%20Multiplications.pdf

It seems that some modern calculators have picked up on some of these methods, particularly the one for inexact square roots, which has now appeared in simplified form in a couple of other books.

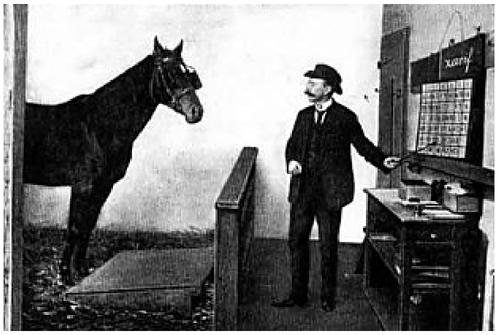
This, then, represents a short summary of some of the methods that have been developed for mental calculation. Again it is important to realize that lightning calculators historically developed highly individualized ways of doing things, and many of those ways were fairly inefficient. But optimum efficiency was not necessarily critical, particularly considering the lack of objectivity among those reporting the exploits of these individuals. To grasp the true history of lightning calculators and their art it is important to recognize this media partiality, and this is the subject of the next part of this essay.

Part III: The Media

Mental calculators of yesteryear were usually described in magazines, newspapers and books in ways that can be startling in our more cynical age. But even today newspaper articles, documentaries and television features on modern lightning calculators appear almost regularly, often with a "hook" such as diminished capabilities in other areas (the "Einstein" effect). Surely there must be some reports that try to be objective, but I haven't found them. At best they are naively written by people with little mathematical background; at worst they use considerable license (deception, really, if only by omission) to present a better story. This part of the essay is not directly related to the historical art of mental calculation itself, but I think it serves as a cautionary tale in evaluating articles on it.

Hans the Clever Horse

Let's take a quick look at a historical example of media misrepresentation, in this case an unintentional one. In the late 1800s and early 1900s the horse shown in the pictures here (Clever Hans) was thought to have the ability to perform arithmetic as well as other reasoning tasks expressed by tapping a hoof a certain number of times. The New York Times wrote a feature on the horse in 1904 (*BERLIN'S WONDERFUL HORSE; He Can Do Almost Everything but Talk—How He Was Taught*) that brought enough attention to the matter that the German board of education created a team of experts to investigate the situation. Following extensive testing, the



New York Times reported (correctly) that the committee had found no evidence of trickery and concluded the horse was exhibiting genuine skills. Only later did the psychologist Oskar Pfungst conduct enough blind tests to determine that the horse was reacting to unconscious cues by the questioners. (For a really good read on this, see http://www.damninteresting.com/?p=384)

So you can't always believe what you see or read, and more generally it takes a critical look (and maybe some cynicism) to separate the chaff from the wheat. And when the story is worth retelling, and particularly when the calculator is a savant, it's often difficult to be objective about the subject.

Innocent Sources of Hyperbole

Often information for newspaper articles is taken from promotional material or verbal descriptions by biased acquaintances or naïve observers, and of course there is always the temptation to embellish the truth a bit. There are accounts I've read of confederates in the audience asking questions or seeding problems with numbers having special properties that make, say, multiplication or division with another genuinely produced number much easier. This isn't unlikely if you think of the "showman" type of lightning calculators, ones who mix these demonstrations with mentalism or magic (Arthur Benjamin, however, is a true lightning calculator as well as a magician). I can also say that the few times I have seen a mental calculator in action, the audience cannot distinguish between presentations of pure mental calculation and simple, standard ways of completing a magic square, for example, that I think of as dross. I also think it's fair to say that incorrect answers are seldom reported, particularly if the calculator corrects the error.

prestidigitator. He never made a mis-
take in his sums, and perhaps he made

Sometimes the problems posed by honest people turn out to be simple for the calculator given their extensive practice. For example, Smith records a number of questions that involve the number of seconds in some number of years; hours in some months, days and hours; cubic yards in some cubic miles; and so on for various simple multiples of common unit conversions known by any calculator of the time.

of His Mental Powers.

Special to The New York Times.

TRENTON, N. J., April 12.—William Vallance, the famous lightning calculator, who could do any sum in mathematical calculation mentally and with but an instant's hesitation, died here to-night, aged 30 years. About a week ago he was taken to the State Hospital suffering from a severe mental strain, believed to be the result of his juggling with figures.

Vallance could duplicate the feats of any of the lightning calculators, and then beat them all by stating instantly any desired date in history. His mind was a vast

It also happens that the calculator gets lucky in a problem selection or in an answer, and then it's one for the record books. Although I'm not a lightning calculator by any stretch, I've certainly benefited from a lucky guess. There was a particularly complicated calculation in my physics class once, involving many terms with powers in the numerator and denominator. As I was wont to do while students reached for their calculators, I wrote down what I thought the first few digits were, which was actually a stretch for me given the problem, and then just wrote two more digits randomly. When the first student read out the answer from the calculator only the last digit was off, and only by 1. There was absolute silence in that classroom as I turned and changed that last digit, and I saw some interesting looks when I turned back around, but of course I never said a word about it.

People sometimes stumble by providing a problem they think is difficult, such as choosing odd numbers or even prime numbers, without realizing that the particular numbers offer a convenient shortcut or, more likely, that the calculator has already memorized the results for these numbers. We will see later that in a documentary on Tammet the researchers decided to ask him a high power of a 2-digit number. Were they going to pick an even number, or maybe one in which the digits were the same? Not likely—primes seem ideal, and there are limited numbers of them. In fact the limits of the calculator display and the powers they selected limited the number to less than 40, so how many likely numbers are there? (he was asked for 37^4, 27^7 and 31^6 in the documentary, and it is true that 27 is not prime). I can't claim that Tammet knew these, but he may have at least known some intermediate powers of these that might have helped (and Tammet has prodigious powers of memory for numbers). This is fine and fair game for any lightning calculator in my opinion. Klein, for example, knew a wealth of number facts, such as "the first 32 powers of 2, the first 20 powers of 3, and so on." In fact, in referring to Dase's calculational efforts, Gauss wrote, "One must distinguish two things here; an extraordinary memory for numbers and true calculating ability. These are, in fact, two completely separate quantities that may be connected, but are not always."

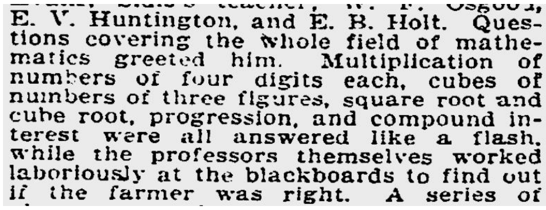
could do sums in his head with a rapidity which seemed incredible, in a flash,

not a student. He could not tell how he knew history, but would rattle off fact after fact without ever making a mistake. He could give instant answer to such arithmetical questions as multiply 389,487 by 4,641. Feats in algebra were his de-

And it's easy to read through an account and unthinkingly accept the writer's assumptions. When I was young I read an account in which the interviewer wrote down a 20-digit number on a napkin and presented it to a memory expert for 15 or 30 seconds, after which the person could read it backwards and forwards. I realized that I could certainly do that, and a lot of people can, say by mnemonics or by grouping it into five 4-digit numbers. In tests by Binet, Diamandi was able to memorize on average 11 digits in 3 sec, 16 digits in 5 sec, and 17 digits in 6 sec, although Binet indicates a significant error rate. Is this so tough? After all, 3 seconds is really a longer span of time than you might think. At Eberstark's request, Smith tested him by reading aloud single digits at a tempo specified by the calculator, about 1.75 sec between digits, for 20 digits, (Eberstark at the end extended this to 40 digits). Is this hard? There are those who do in fact perform amazing feats of quick memorization (in tests Salo Finkelstein repeated a 20-digit and 25-digit number after exposure for 1 sec apiece, a 33-digit number exposed for 2 sec, and 39 digits exposed for 4 sec), but the lesson here is to be critical when reading articles or watching programs.

As a final example, in a 2005 performance in the second Arthur Benjamin video listed earlier at http://www.youtube.com/watch?v=M4vqr3_ROIk, four audience members were brought on stage at the start to verify his answers on calculators. Benjamin did a variety of calculations, most of them correctly, but it's interesting that despite his turning to them to request verification, two of his five answers in squaring 3-digit numbers were incorrect. But none of the four challenged his answers, and to be honest, I wouldn't have had enough confidence that I entered the digits correctly to have held up a show like that either. So don't trust observers *or* judges.

Conscious Bias in Reporting


E. V. Huntington, and E. B. Holt. Questions covering the whole field of mathematics greeted him. Multiplication of numbers of four digits each, cubes of numbers of three figures, square root and cube root, progression, and compound interest were all answered like a flash, while the professors themselves worked laboriously at the blackboards to find out if the farmer was right. A series of

Sometimes the writer or director purposely skews the reporting in ways that are probably conscious but not that serious—sins of omission and that sort of thing in a light piece. For those with an interest in the subject beyond casual reading, it's important to notice these nuances.

Smith's book is rife with contemporary accounts that use phrases such as "in an instant" or "in a flash" or "in the blink of an eye" and so forth. And the details of the task are seldom presented, even in structured tests by researchers, a fact that absolutely amazes me. Did the calculator repeat the problem back to the questioner? Did the questioner write the question down in front of the calculator, did the calculator have the problem in view during the test, and was timing (if there was any) stopped when the first digit of the answer was being written or the last, or when the calculator said "Done" or when the last digit was recited? And we have seen that there are particular types of problems (e.g., the number of seconds in a given number of years) that benefit hugely from memorized facts. Smith also points out instances in which the set of test questions by researchers all shared the same shortcut property—why is that? How many questions were asked in all? Were just the correct, speedy ones reported? One rarely if ever has these facts in an account of a lightning calculator.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From The Times

April 8, 2005

What is the 13th root of…

83689566882369569398373286622256452247267804664938366774973575581573035075704089625288023857831568376802934938201056343363855595931514450415149490709419097704449305660268402771869624155688082648640933?

From Charles Bremner in Paris

Under the eye of official timekeepers, Alexis Lemaire, 24, a computer science student from Reims, took 513 seconds to find the 16-digit number which, multiplied by itself 13 times, produced a 200-digit sequence that had been presented to him on a

This continues today, of course. Let's look at a common example of subtly slanted reporting. You might think I'm seeing bias where there is none, but when you read enough of these accounts you begin to see patterns. Alexis Lemaire is an extremely talented mental calculator with a specialty of extracting 13^{th} roots of 200-digit numbers, a feat that I don't believe is attempted by others. So my comments here are only on the reporting and in no way reflect on Mr. Lemaire or his abilities.

So here's a typical news report of a record time set by Lemaire at the Oxford Museum of the History of Science, dated July 30, 2007, by BBC News and found at http://news.bbc.co.uk/2/hi/uk_news/magazine/6913236.stm .

> The task is to find the 13th root of 85,877,066,894,718,045,602,549,144,850,158,599,202,771,247,748,960,878,023,151,390,314,284,284,465,842,798,373,290,242,826,571,823,153,045,030,300,932,591,615,405,929,429,773,640,895,967,991,430,381,763,526,613,357,308,674,592,650,724,521,841,103,664,923,661,204,223.
>
> The answer's 2396232838850303. Multiply that by itself 13 times and you get the above. Even with a calculator you wouldn't beat Alexis Lemaire doing the calculation in his head.

Another article from 2005 on such a record, shown in the figure above, can be found at http://www.timesonline.co.uk/tol/news/world/article378484.ece .

Now let’s look at this article of another such record from any of the seven reports I found very quickly online via Google. For direct comparison with the first one above I’ll choose BBC News again, dated December 11, 2007, and found at http://news.bbc.co.uk/2/hi/uk_news/england/london/7138252.stm .

> The fastest human calculator has broken his own mental arithmetic world record.
>
> Alexis Lemaire used brain power alone to work out the answer to the 13th root of a random 200-digit number in 70.2 seconds at London's Science Museum.
>
> The 27-year-old student correctly calculated an answer of 2,407,899,893,032,210, beating his record of 72.4 seconds, set in 2004.
>
> The so-called 'mathlete' used a computer package to randomly generate a number before typing in the answer.

So here we are given a little more information on how the test was performed. But do you see the real difference here? The randomly generated number is not reported, just the root. Why is that—generally the huge number is much more impressive to present than the root. Well, let’s see what that randomly generated number was:

> 9147439728147451289480367741620143028356421050343238533956132727693345422960930
> 4646471925094518114771016258896592907441426349897556504145570960203925503679105
> 24519914233880608249425405061000000000000

It doesn’t look so random. In fact, it wasn’t the power that was randomly generated (after all, what are the odds that a randomly generated number would be a 13th power?), but rather the root was randomly generated and the power calculated from that value. Which is fine, but it seems apparent to me, at least, that they would have reported the power if it didn’t end in thirteen zeros. The reader might immediately intuit that last digit of the root is 0, so it detracts slightly from the effect and makes them consider that it was a lucky break. In fact the last digit is always identical in a number and its 13th power so it’s always trivial to find it, but this now makes the second-to-last digit trivial (the digit 1). I see a little bias on the part of the reporter, and I saw this in every report of this event I could find.

Let’s take a news report of another record-breaking event from November 16, 2007, found at http://www.shortnews.com/start.cfm?id=66565

> 27-year-old Alexis Lemaire from France has set a new world record by mentally calculating the 13th root of a 200-digit number in 72.4 seconds. He correctly identified the answer as 2,397,207,667,966,701. The previous record was 77 seconds.

No 13th power listed here either. And I believe the reason is that this power is

> 8633234880035284361012699002231346851047737093075599215268139034779532300
> 9751168717005763648080727141383324712170576311110855841562345802001852566
> 12852897226196105357173387251523920946707380414694987101

With the power and root in view it doesn’t take too long to figure out that any power ending in 01 would have roots ending in 01, so this was again a case in which the last two digits are found instantly. This may not have been a lot of help to Lemaire, as he probably knows all two digit endings of 13th roots, but again this is all about the reporting. Also, these are situations where it’s easy for us to see the advantages of a particular number, whereas lightning calculators have a wealth of stored number facts that can make certain problems much easier in a less apparent way, and it only takes one lucky number to break a record.

Deconstructing the *BrainMan* Documentary

Let's take a look at a documentary that in my opinion inadvertently reveals itself as duplicitous. This may seem a bit tedious, but I'm kind of proud of the mathematical detective work I did in uncovering this.

Daniel Tammet holds the world record for memorizing pi (22,514 digits in all!), he has appeared on *60 Minutes* and David Letterman's show, and he has written a recent autobiography titled *Born on a Blue Day: Inside the Extraordinary Mind of an Autistic Savant* (see http://www.optimnem.co.uk/book.php). He has some talent with mental calculation as well, and he was featured in a popular 2004 documentary called *BrainMan* (titled *The Boy With The Incredible Brain* in the UK). The documentary won a Royal Television Society award in December, 2005, and was nominated for a BAFTA in 2006.

Tammet experiences synaesthesia, the ability to see or experience numbers as shapes, colors and textures. Here's a typical excerpt from an article on him:

> Tammet is calculating 377 multiplied by 795. Actually, he isn't "calculating": there is nothing conscious about what he is doing. He arrives at the answer instantly. Since his epileptic fit, he has been able to see numbers as shapes, colours and textures. The number two, for instance, is a motion, and five is a clap of thunder. "When I multiply numbers together, I see two shapes. The image starts to change and evolve, and a third shape emerges. That's the answer. It's mental imagery. It's like maths without having to think."

Now I was asked awhile back about Tammet's solution of 13/97 in the *BrainMan* documentary. I had not seen it, but I replied that division by a two-digit number like 97 is not difficult (130/97=1 remainder 33, 330/97=3 remainder 30+3(3)=39, 390/97=4 remainder 2, etc., so we get .134... and so on---lightning calculators can fly through this). However, we saw earlier here than the reciprocal of 97 consists of repeating groups of 96 digits. I thought it likely that either half the repeating group or all the repeating group of 1/97 was memorized, because changing the numerator to any 2-digit number less than 97 simply cycles the starting position of the repeating group to another location. In fact, Aitken had remarked on the commonly proposed problem of this reciprocal in his talk found at http://stepanov.lk.net/mnemo/aitkene.html :

> Here the remark was made that memory and calculation were sometimes almost indistinguishable to the calculator. This was illustrated by the recitation of the 96 digits of the recurring period of the decimal for 1/97, checked by Dr. Taylor. Probably because 97 was the largest prime number less than 100, this particular example had been frequently proposed.

Actually, I suspect division by 97 is often asked because it takes a whole 96 digits before the digits start repeating.

Later that day it occurred to me that there might be a way to detect whether that problem was solved by Tammet through a memorized repeating group. The reciprocal of 97 is

1/97 =
.010309278350515463917525773195876288659793814432|
989690721649484536082442268041237113402061855 67|
010309278350515463917525773195876288659793814432|
989690721649484536082442268041237113402061855 67|
01030927...

and also

13/97 =
.134020618556701030927835051546391752577319587628|
865979381443298969072164948453608247422680412371|
134020618556701030927835051546391752577319587628|
865979381443298969072164948 453608247422680412371|
13402062...

I've added vertical bars at each half of the 96-digit repeating group. We can see that each decimal expansion starts repeating every 96 digits. In addition, each digit in one half of the repeating group is the difference from 9 of the corresponding digit in the other half of the repeating group as mentioned earlier in the *Fast Division* section of this essay. So to produce 1/97 we can just memorize the first 48 digits of the repeating group, and then repeat that it but subtract each digit from 9. Again, Aitken had done that in anticipation of being asked for it in performances.

You can see that 13/97 has simply cycled the repeating group to the position near the end that starts with 134. The starting point for a given numerator is not predictable, but we can just divide 13 by 97 to a few digits (.134) to find the start, or we can multiply 13 by the first few digits of the repeating group for 1/97 (.0103) to find 134 as the starting point. If only the first half of the repeating group of 1/97 is memorized, we would see if 134 is in that group, and if it isn't there (like now) we look for the 9's complement 865 in that group, which is found near the end. So we start at that point, listing the 9's complement of each digit as we cycle around the start of that half-group, and when we reach 865 again we repeat the steps but don't take the 9's complement. So we can always get away with memorizing just 48 digits to divide any number by 97. If the numerator is greater than 97, it's just a whole number and a fraction with a numerator less than 97, so we end up being able to divide any number at all by 97 this way. (Technically speaking, we only need to memorize 47 digits because the repeating group for division by any number ending in 7 ends in 7, but then we'd have to remember that fact.).

Well, I thought that given the different starting position of 13/97, if Tammet were using a memorized half or full repeating group of 1/97, a verbal hesitation might be detectable at the end of this group as he "resets" his memorized or mnemonic digits to the start of this group. The 1/97 repeating group ends with the digits 567 and then cycles back to the beginning to 010... This 567 sequence occurs in the 11th to 13th position in 13/97. If Tammet hesitated between stating the 7 in the 13th position and the 0 in the 14th position in reciting 13/97, it would be evidence that the memorized group (or half-group) of 1/97 was being utilized.

So after all of this I finally watched the documentary online. You can find the complete UK version of the documentary in 5 parts on YouTube. The first part is at http://www.youtube.com/watch?v=AbASOcqc1Ss . Google Video has the U.S. version in a single video, but it does not include the scene with the whiteboard that I will refer to below.

The 13/97 calculation is not far into the documentary (at 1:49 in Part 1), so it's quite interesting to watch all of it up to that point to see how the shapes/crystallization process is described. He actually does the 13/97 calculation twice back-to-back, it turns out.

We don't hear the problem posed to Tammet in the video, so we don't know the delay between when the question was asked and the response was begun. Tammet slowly recites the digits of the answer all the way up to 5..6..7 and then—he comes to a total halt! He has lost his place, the questioner mentions that he is carrying on, he says he's carrying on, then he says to tell him to stop or ... and the interviewer stops him. I about fell off my chair. The interviewer asks him how many places he can do it to, and Tammet replies, "A hundred—nearly a hundred." To me this reveals that he has done division by 97 before, and as we know if you can do 96 digits you can repeat them as long as you want to.

And this brings me to something very misleading---that the interviewer implies here that division gets harder as you get further into the solution. This is absolutely false—in division it's just as easy to get the 1,000th digit as it is to get the 2nd digit. It is not like, say, a square root.

And I'm in luck—Tammet is going to repeat his answer after the interviewer retrieves a computer to get more digits. And here I will say that anyone who digs up an old 8-digit calculator to go test a savant on his calculating abilities, especially on division, can only be setting up a dramatic scene in a fluff piece. And another thing: the interviewer tells Tammet to wait, they are going to go find a computer, then come back, boot it up, sync the camera to the vertical sync of the video card, run the calculator application, and ask him the very same problem to see how many digits he can do?? And what would someone like me be doing the whole time they're fiddling around? I'd be furiously calculating more digits in my head, that's what.

But Tammet starts reciting again at about the same rate as the video scans the digits on the computer screen. He gets to the 5..6..7... and to my astonishment at that exact moment (and I mean *exactly* at the instant the next digit would be uttered) a voiceover is spliced into the video, saying two superfluous things—that every digit is correct (which we would know if the voiceover wasn't there) and that he will eventually exceed the 32 digits of the computer (which we find out when we get there). Also, at this same point a video cut is shown of his hands making movements on the table. Then the audio and video return to him reciting about a dozen places later. If you replay the video and you continue reciting the correct digits during the voiceover, you find that Tammet would have had to have sped up significantly to have been at that point when they return to his recitation, although to be fair he does speed up at the very end of it all.

So it strongly appears to me that the documentary covered up for him on what I think must have been some sort of difficulty at the point I predicted. Not a big deal, maybe, but the producers went to a lot of trouble to deceive us on this, and that's makes me question the validity of the whole enterprise.

In Tammet's book on page 4 he says he "calculates" divisions like 13/97 by seeing spirals rotating in loops that seem to warp and curve, and in fact if you go to nearly the end of Part 4 of the documentary (from 9:41 to 9:44 in http://www.youtube.com/watch?v=UqLzoiVzEY8) and look carefully when Tammet is tracing such a spiral on the whiteboard (see the frame capture here), you'll find it is being drawn right below "1/97". So this divisor pops up again, lending credence to the theory that it was half- or fully-memorized.

Finally, let's look at the three integer powers that are asked of him. Very early in Part 1 (at 0:58 in http://www.youtube.com/watch?v=AbASOcqc1Ss) the narrator says that Tammet was asked to find 37^4. We don't see it asked, we just see the interviewer punching 37 x 37 x 37 x 37 extremely slowly into the calculator, followed by a continuous pan to Tammet, who looks up and recites the answer. So the question was asked at some unknown time prior to the entry into the calculator.

Later in the Part 4 of the documentary (at 0:40 in http://www.youtube.com/watch?v=UqLzoiVzEY8) Tammet is asked to find 27^7 and 31^6 in two apparently unplanned, poorly executed tests by two neuroscientists (are there no x^y keys on these calculators??). We'll never know how long it took to find the results because the documentary has so many cuts injected there that our sense of time is destroyed while the background music gives a false sense of continuity. We do see that they don't start a small timer until 4 seconds after one of the problems is given. Again, the producers of the documentary mislead the audience by compressing the timescale. And for those who still might have thought the documentary to be unbiased, a voiceover appears during the latter calculation to blame whatever delays there were (what were they?) on jet lag.

So to summarize all this, in the process of trying to analyze Tammet's method I found strong evidence that the *BrainMan* documentary in several ways actively misled the viewers. And of course this all has to do with the producers of the documentary themselves, not Daniel Tammet. And that's why you have to be critical of these sorts of things.

Now let's consider the first part of the documentary listed earlier on Rüdiger Gamm at http://www.youtube.com/watch?v=NUsD2V6ijyQ&feature=related . Very early into it, just after 0:40 sec, Gamm announces to an auditorium that he will attempt to divide the prime number 109 into a 2-digit number provided by an audience member. He will attempt to go 100 digits after the decimal place. After receiving a number of 93, Gamm repeats the problem "93/109" and focuses on the problem for a total of 11 sec. Then he starts reciting the digits, very soon accelerating and reciting the digits as fast as he can say them.

Every alarm in your head should be going off about now. Is the number Gamm chose (109) one of those primes whose reciprocal has the maximum possible repeating group (108 digits)? Did he recite only 100 digits so the repetition after 108 digits wouldn't be noticed? Yes, and in my opinion, yes. Here's the reciprocal of 109 with vertical bars separating halves of the repeating group as in our earlier example for 97:

1/109 =
.009174311926605504587155963302752293577981651376146788|
990825688073394495412844036697247706422018348623853211|
009174311926605504587155963302752293577981651376146788|
990825688073394495412844036697247706422018348623853211|

0091743119…

So the repeating group does have 108 digits, and as always in this case, the digits in the second half are the 9-complements of the corresponding ones in the first half. So let's take the submitted numerator 93 and mentally divide it by 109 to three digits. We can divide 93 by 110 instead and adjust for the offset in each step by adding the previous digit, as presented earlier in the *Fast Division* section for division by a number ending in 9:

93/11 = 8 remainder 5
(50+8)/11 = 5 remainder 3
(30+5)/11 = 3 remainder 2

and we locate 853; it's in the second half of the group near the end. If the repeating group for 109 is memorized (or even half of it as described earlier since the other half is the 9's-complement), it's child's play to recite the digits.

Now I don't know how Gamm actually performed this feat. If you practice just a bit with the adjusted division process you can develop a kind of rhythmic cadence as you go:

93 **8** *58* **5** *35* **3** *23* **2** *12* **1** *11* **0** *10* **0** *100* **9** *19* **1** *81* **7**…

This is remarkably easy if you try it without reading it. Stating the bolded digits out loud really helps to append them to the remainder of the next division. Gamm does seem to develop a sort of cadence in the video, and he is a phenomenal calculator, so it's likely that he is just amazingly fast at this. In any event, to judge the performance it's important to realize that an adjusted division technique exists, and it's also worth noting that with some memorization you too could walk into an auditorium and perform as well as Gamm on this.

The Appeal of the Mental Calculator

The study of lightning calculators of the past is a fascinating one for me from a mathematical aspect more than a psychological one. We've seen years of articles by educators bemoaning the dependence of students on calculators, but I see little in school textbooks on mental math other than simple estimation. And yet when I have presented basic methods of mental calculation to classes (elementary and college), I've met with incredible interest. Certainly the *BrainMan* documentary is a very popular one. But these types of presentation generally ascribe abilities in these areas to mysterious machinations in the minds of remote geniuses, which makes for a good story but can be discouraging. In fact, these individuals through talent and training acquired a knack for racing headlong through calculations that are not mysterious at all once the methods are taught.

And they are not being taught. Mental calculation can be a highly creative and satisfying endeavor offering a variety of interesting strategies, more than I have presented here and many more than most people realize. It is a skill that engages both children and adults, and one that naturally leads to a real familiarity with the properties and relationships of numbers. It provides a useful and fun approach for developing a number sense and generating a true appreciation for the elegance of elementary mathematics. It should not be a neglected art.