Example of the Saint-Robert Criterion

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Can the equation $z = xy + \sqrt{1+x^2}\sqrt{1+y^2}$ be constructed as a parallel-scale nomogram? We will use Saint-Robert's criterion.

Here $F(x,y,z) = -z + xy + \sqrt{1+x^2}\sqrt{1+y^2} = 0$.

$$\frac{\partial F}{\partial x} = y + \sqrt{1+y^2}(1/2)(1+x^2)^{-1/2}(2x) = y + x\sqrt{\frac{1+y^2}{1+x^2}}$$

$$\frac{\partial F}{\partial y} = x + \sqrt{1+x^2}(1/2)(1+y^2)^{-1/2}(2y) = x + y\sqrt{\frac{1+x^2}{1+y^2}}$$

$$R = \frac{\partial F/\partial x}{\partial F/\partial y} = \frac{y + x\sqrt{\frac{1+y^2}{1+x^2}}}{x + y\sqrt{\frac{1+x^2}{1+y^2}}}$$

$$= \frac{\frac{y\sqrt{1+x^2}+x\sqrt{1+y^2}}{\sqrt{1+x^2}}}{\frac{x\sqrt{1+y^2}+y\sqrt{1+x^2}}{\sqrt{1+y^2}}} = \sqrt{\frac{1+y^2}{1+x^2}}$$

$$\ln R = 1/2\ln(1+y^2) - 1/2\ln(1+x^2)$$

$$\frac{\partial \ln R}{\partial x} = -1/2\left(\frac{1}{1+x^2}\right)(2x) = -\frac{x}{1+x^2}$$

$$\frac{\partial^2 \ln R}{\partial x \partial y} = 0$$

This result means we can represent $F(x,y,z) = -z + xy + \sqrt{1+x^2}\sqrt{1+y^2} = 0$ as a parallel-scale nomogram, or in other words, we can express it in the form $Z(z) = X(x) + Y(y)$ or in the form $Z(z) = X(x)Y(y)$ which can be rewritten as $\ln Z(z) = \ln X(x) + \ln Y(y)$.

To find $X(x)$,

$$\ln\frac{dX}{dx} = \int \frac{\partial \ln R}{\partial x}dx = \int -\frac{x}{1+x^2}dx = -1/2\ln(1+x^2) = \ln\left(\frac{1}{\sqrt{1+x^2}}\right)$$

so $$\frac{dX}{dx} = \frac{1}{\sqrt{1+x^2}}$$

from integral tables, $X = \ln(x + \sqrt{1+x^2})$ which is sometimes given as $\sinh^{-1} x$

For $Y(y)$,

$$
\begin{aligned}
\frac{dY}{dy} &= \frac{{}^{dX}\!/_{dx}}{R} \qquad \text{which will contain no variable } x \\
&= \frac{\frac{1}{\sqrt{1+x^2}}}{\sqrt{\frac{1+y^2}{1+x^2}}} = \frac{1}{\sqrt{1+y^2}} \\
\text{so} \qquad Y &= \ln(y+\sqrt{1+y^2})
\end{aligned}
$$

And for $Z(z)$, we can use $Z(z) = X(x) + Y(y)$:

$$
\begin{aligned}
Z(z) &= \ln(x+\sqrt{1+x^2}) + \ln(y+\sqrt{1+y^2}) \\
&= \ln\left[(x+\sqrt{1+x^2})(y+\sqrt{1+y^2})\right]
\end{aligned}
$$

We are guaranteed that we can use $z = xy + \sqrt{1+x^2}\sqrt{1+y^2}$ to express $Z(z)$ in terms of z and eliminate all x and y terms. With some algebra we can find that

$$Z(z) = \ln(z+\sqrt{z^2-1})$$

Substituting X, Y and Z into $Z(z) = X(x) + Y(y)$ we have

$$
\begin{aligned}
\ln(z+\sqrt{z^2-1}) &= \ln(x+\sqrt{1+x^2}) + \ln(y+\sqrt{1+y^2}) \\
\text{or} \qquad (z+\sqrt{z^2-1}) &= (x+\sqrt{1+x^2})(y+\sqrt{1+y^2})
\end{aligned}
$$

which is the form for a nomogram consisting of three parallel scales.

It turns out that this works when x in the original equation $F(x,y,z)$ is replaced with any function of x, y is replaced with any function of y, and z is replaced with any function of z. It is a surprising result, and one that Maurice d'Ocagne included in his 1899 book, *Traité de Nomographie* (pages 418-421).