The McIntosh-Doerfler Algorithm for Exponentials

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We will provide examples that follow the McIntosh-Doerfler algorithm presented at

http://www.urticator.net/essay/6/641.html

to find the value of 10^t for a 5-digit value of t.

Most algorithms to find exponentials (such as the Bemer method), require memorization of a set of logarithms to choose to subtract from the original exponent to get the remainder "x" as near to 0 as possible. Then the first two or three terms of the relation $e^x = 1 + x + x^2 / 2 + \ldots$ is used to find e^x for the remaining small x, or since $10^x = e\,(2.303\ldots)x$, then $10^x = 1 + 2.303x + (2.303x)^2/2$ is used. For a negative value of x, the value of x is made positive by adding 1 and the equation $10^x = 1 - 2.303x + (2.303x)^2/2$ is used instead. Note that for 10^x, 1 can be added or subtracted from the exponent by multiplying or dividing the result by 10, so a value of x near 1 or -1 is also feasible in the latter case.

The advantage of the McIntosh-Doerfler algorithm is that only the values of log 3 and log 2 need to be memorized. Actually, since only one or two multiples of log 3 are needed at most, it is useful to memorize both log 3 and twice that value. As for log 2, this turns out to be a simple number to find multiples of since log 2 = 0.30103 has simple digits that don't have multiplication carries between non-zero digits. This is a very convenient feature that does not appear in natural logs of low integers.

This method also emphasizes the advantage of "wrapping" values around 1 to remove powers of 10 to simplify intermediate results, or taking second complements to turn negative numbers into positive numbers. These wraps do not need to be remembered, because the range of the answer is found from the original problem after the final value is calculated, and the decimal point of the final value is then shifted to put it into the correct range.

First, a definition: the **second complement** of a number is found by subtracting each digit of a number from 9 and then adding 1 to the result. For a decimal number < 1, this provides the decimal digits when 1 is added or subtracted from that number, or in other words, when the value is wrapped around 1. So for example, -0.95424 + 1 = 0.04576 where each digit is subtracted from 9 and then 1 is added (or 0.04575 + 0.00001).

The Method

Assuming a 5-digit value of t, the steps of the algorithm to find 10^t are:

1. Ignore for now any integer part of t. These are simply whole powers of 10. The remaining decimal part, still assumed here to be 5 digits long, can also have an additional integer 1 added or subtracted, so in general the decimal value can wrap around integer 1 in either direction. If the value is negative, find the second complement to make it positive, so -0.95424 + 1 = 0.04576. Note that we do not have to recall how many times we wrap the value around 1 because we will simply get the final range at the end by looking at the original exponent.

2. Add or subtract up to two copies of log 3 = .47712 to make the number close to a multiple of 0.1. You can determine the right number of copies by looking at the second and third digits of the number because adding log 3 is like subtracting 23 in those digits, and vice versa.

3. Add or subtract up to five copies of log 2 = .30103 to make the number close to zero, or -1 or 1 by realizing that you can wrap the result around 1. You can determine the right number of copies by looking at the (rounded) first digit and seeing what multiple of 3 will cancel it out. If the number is .39700, for example, then the rounded first digit is 4, which can be canceled by adding 2×3 = 6 = 10, so therefore the right move is to add two copies of log 2.

4. Call the current number x, and evaluate the formula $1 + 2.303x + (2.303x)^2/2$. Note that you take the value 2.303x and square it and divide by 2 to find the last term. The squared term only needs two or three digits. If x is close to zero on the .99999 side, take the second complement and remember that at the end you'll need to subtract the 2.303x term instead of adding it. Also, if you don't need all five decimal places, you can use three places instead, and evaluate the simpler formula 1 + 2.3x. That variant is extremely fast, and quite accurate too, as far as it goes.

5. For every log 3 that was subtracted in step 2, multiply the result of step 4 by 3, or for every log 3 that was added, divide the result by 3. For every log 2 that was subtracted in step 3, multiply the result by 2, or for every log 2 that was added, divide the result by 2.

6. Shift the decimal point to bring the result into the correct range. The range is determined looking at the original problem, so for example $10^{2.40000}$ lies between 10^2 and 10^3.

Here are the three examples listed on the webpage above in greater detail that shown on that webpage.

Problem A: Find $10^{0.20000}$

1. There is no integer part of 0.20000 to subtract and no reason to add an integer 1.
2. There is no need to subtract one or two values of log 3 to make it close to a multiple of 0.1 since it already is a multiple of 0.1.
3. How many multiples of log 2 = 0.30103 to add or subtract to get 0.20000 close to 0 (or 1 since we can wrap around 1)? The rounded first digit of 0.20000 is 2, and if we subtract 4x3 we would get -10. So we find 0.20000 – 4 x 0.30103 = 0.20000 – 1.20412 = -1.00412. Drop the whole value -1 and convert the rest to a positive value by taking the second complement, so we get 0.99588.
4. Here the value is just under 1 so we will again take the second complement x = 0.00412 and therefore remember to now subtract the second term of the correction rather than adding it, so we add $1 - 2.303x + (2.303x)^2/2$. First, find 2.303 x 0.00412 = 0.00949 for the second term. For the third term, square 0.00949 and divide by 2 to get 0.00009/2 = 0.00005. So 1 - 0.00949 + 0.00005 = 0.99056.
5. We subtracted 4 copies of log 2 in step 3, so we multiply 0.99056 by 2^4 = 15.84896.

6. Looking at the original value $10^{0.20000}$, we can see that the result lies between 10^0 = 1 and 10^1 = 10, so we shift the decimal point to get 1.58490 as our answer. The actual value is 1.58489.

Problem B: Find $10^{3.14159}$

1. Subtract the integer part of 3.14159 to get 0.14159.
2. Add or subtract up to two copies of log 3 = .47712 to make the number close to a multiple of 0.1. Adding log 3 is like subtracting 23 in the second and third digits of the number, and vice versa. These digits are 41, so we can add two log 3 values to get these digits near zero, since 41 - 2 x 23 = -5. So 0.14159 + 2 x .47712 = 1.09583, which after dropping the integer 1 is 0.09583.
3. How many multiples of log 2 = 0.30103 to add or subtract to get 0.09583 close to 0 (or 1 since we can wrap around 1)? The rounded first digit of 0.09583 is 1, and if we add 3x3 we would get 10 so we add three copies of log 2. So we find 0.09583 + 3 x 0.30103 = 0.09583 + .90309 = 0.99892.
4. Here the value is just under 1 so we will take the second complement x = 0.00108 and therefore remember to now subtract the second term of the correction rather than adding it, so we add 1 - 2.303x + $(2.303x)^2/2$. First, find 2.303 x 0.00108 = 0.00249 for the second term. For the third term, square 0.00249 and divide by 2 to get 0.00001/2 = 0.00000. So 1 - 0.00249 + 0.00000 = 0.99751.
5. We added two copies of log 3 and added three copies of log 2 in step 3, so we divide 0.99751 by (3^2 * 2^3) = 0.99751 / 72 = 0.0138543.
6. Looking at the original value $10^{3.14159}$, we can see that the result lies between 1000 and 10000, so we shift the decimal point to get 1385.43 as our answer. The actual value is 1385.45.

Problem C: Find $10^{-0.33550}$

1. There is no integer part to subtract off. Since the exponent is negative, take the second complement to get 0.66450.
2. Add or subtract up to two copies of log 3 = .47712 to make the number close to a multiple of 0.1. Adding log 3 is like subtracting 23 in the second and third digits of the number, and vice versa. These digits are 64, so we can subtract two log 3 values to get these digits near zero, since 64 + 2 x 23 = 110. So 0.66450 - 2 x .47712 = -0.28974, which we add 1 to get 0.71026 which is somewhat near a multiple of 0.1.
3. How many multiples of log 2 = 0.30103 to add or subtract to get 0.71026 close to 0 (or 1 since we can wrap around 1)? The rounded first digit of 0.71026 is 7, and if we add 1x3 we would get 10 so we add one copy of log 2. So we find 0.71026 + 1 x 0.30103 = 1.01129 that we subtract 1 to get 0.01129.
4. Here x = 0.01129. We add 1 + 2.303x + $(2.303x)^2/2$. First, find 2.303 x 0.01129 = 0.02600 for the second term. For the third term, square 0.02600 and divide by 2 to get 0.00068/2 = 0.00034. So 1 + 0.02600 + 0.00034 = 1.02634.
5. We subtracted two copies of log 3 and added one copy of log 2, so we multiply 1.02634 by 3^2 and divide by 2. This gives 1.02634 x 9/2 = 4.61853.

6. Looking at the original value $10^{-0.33550}$, we can see that the result lies between 0.1 and 1, so we shift the decimal point to get 0.461853 as our answer. The actual value is 0.461849. We knew this wasn't going to have as good an accuracy as the other examples because x wasn't as close to zero.